Math Diversion Problem 505

P. Reany

April 8, 2025

It is said that honor dies where interest lies.
— Kung Fu, "Arrogant Dragon"

Learning is a treasure that will follow
its owner everywhere.
— Chinese proverb

The problem is found at:

```
Source: https://www.youtube.com/watch?v=zoytksx8dPE
Title: A Nice Problem With Reciprocals | Problem 466
 Presenter:  aplusbi
```

1 The Problem

Given the relation

$$z^2 - z + 1 = 0\,, \tag{1}$$

find the value of

$$\phi = z^5 + z^{-5}\,. \tag{2}$$

2 The Solution

The first thing I want to do with the Given relation is to obtain

$$z + z^{-1} = 1\,. \tag{3}$$

Next, I want to take the third and fifth powers of this last equation and simplify them:

$$(z + z^{-1})^3 = 1\,, \tag{4}$$

$$(z + z^{-1})^5 = 1\,, \tag{5}$$

which simplify down to

$$z^3 + z^{-3} = -2\,, \tag{6}$$

$$z^5 + z^{-5} + 5(z^3 + z^{-3}) + 10(z + z^{-1}) = 1\,. \tag{7}$$

This last equation can be rewritten as

$$\phi + 5(-2) + 10(1) = 1\,, \tag{8}$$

and so finally,

$$\phi = 1\,. \tag{9}$$