

Math Diversion Problem 507

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Equations are just the boring part of mathematics. I attempt
to see things in terms of geometry.
— Stephen Hawking

The problem is found at:

Source: <https://www.youtube.com/watch?v=kiXOL0niuHM>

Title: Find $\log_x(y)$

Presenter: Prime Newtons

1 The Problem

Given the relations

$$x^{\log_y z} = 2, \tag{1a}$$

$$y^{\log_z x} = 4, \tag{1b}$$

$$z^{\log_x y} = 8, \tag{1c}$$

where $x, y, z > 1$, find the value of $\log_x y$.

2 The Preparation

$$\log_a b = \frac{1}{\log_b a}. \tag{2}$$

And

$$\log_a b \log_b c = \log_a c. \tag{3}$$

3 The Solution

Let

$$\gamma = \log_x y. \tag{4}$$

From (1a),

$$\log_y z \log_z x = \log_z 2. \quad (5)$$

Using (3) and (2), this gives us

$$\log_y x = \gamma^{-1} = \log_z 2. \quad (6)$$

From (1c),

$$\log_x y = \log_z 8 = 3 \log_z 2, \quad (7)$$

or

$$\gamma = \log_z 8 = 3 \log_z 2. \quad (8)$$

Now, we divide (8) by (6), to get rid of $\log_z 2$:

$$\frac{\gamma}{\gamma^{-1}} = \frac{3 \log_z 2}{\log_z 2} = 3. \quad (9)$$

And this simplifies to

$$\gamma = \sqrt{3}. \quad (10)$$

Comment: I did not use all the given information. What do you think of that?
(I did not use (1b).)