Math Diversion Problem 511

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CONCERNING HORSES

Hast thou given the horse strength? hast thou
clothed his neck with thunder?
Canst thou make him afraid as a grasshopper?
the glory of his nostrils is terrible.
He paweth in the valley, and rejoiceth in his strength:
he goeth on to meet the armed men.
He mocketh at fear, and is not affrighted; neither
turneth he back from the sword.
— Job 39

The YouTube video is found at:

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Title: A Lambert Integration
 Presenter: Patrick
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1 The Problem

Do the following indefinite integral:

$$I(x) = \int x W(x) dx \,, \tag{1}$$

where $W(x)$ is the Lambert W function.

2 The Preparation

Some useful things to know:

Concerning the Lambert W function:

$$W(xe^x) = x \,, \tag{2a}$$

$$W(x)e^{W(x)} = x \,. \tag{2b}$$

Concerning useful integrals:

$$\int x^2 e^{ax} dx = \left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3} \right) e^{ax} , \tag{3a}$$

$$\int x^3 e^x dx = \left(x^3 - 3x^2 + 6x - 6 \right) e^x . \tag{3b}$$

3 The Solution

We have two possible ways to proceed. The first is to convert the x's into functions of W; the second is to do it the other way around. I think I'll do it the first way.

From (2b), we have that

$$dx = e^W (1 + W) dW . \tag{4}$$

Using this last result and (2b), (1) becomes

$$I(W) = \int (W^3 + W^2) e^{2W} dW = I_1 + I_2 , \tag{5}$$

where

$$I_1 \equiv \int W^2 e^{2W} dW \quad \text{and} \quad I_2 \equiv \int W^3 e^{2W} dW . \tag{6}$$

I_1 is easy. From (3a), we get

$$I_1 = \frac{1}{8} (4W^2 - 4W + 2) e^{2W} . \tag{7}$$

Replacing x in (3b) with $2W$ and simplifying, we get

$$I_2 = \frac{1}{8} (4W^3 - 6W^2 + 6W - 3) e^{2W} . \tag{8}$$

So, then,

$$I(W) = \frac{1}{8} (4W^3 - 2W^2 + 2W - 1) e^{2W} + C . \tag{9}$$

I've also seen it represented this way:

$$I(W) = \frac{1}{8} (2W - 1)(2W^2 + 1) e^{2W} + C . \tag{10}$$