Math Diversion Problem 514

P. Reany

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It is a capital mistake to theorize
in advance of the facts.
— Sherlock Holmes (Jeremy Brett)
[Episode *The Second Stain*]

The problem is found at:

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Source: https://www.youtube.com/watch?v=B-EIkGzedhE
Title: Deriving Wien's law
 Presenter:  Physics and Math Lectures
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1 The Problem

Given the relation

$$\frac{xe^x}{e^x - 1} = 5\,, \tag{1}$$

find the values of x.

2 The Preparation

The Physics

Wien's law refers to a 'law' that tells us the wavelength of maximum intensity of radiation emitted by a black body.[1] This law was formulated first by physicist Wilhelm Wien in 1893 on the basis of his experimental results.

Deriving Wien's law using Planck's distribution for black body radiation. Planck's distribution law for black body radiation is the following:[2]

$$u_\lambda(\lambda, T) = \frac{2hc^2}{\lambda^5}\frac{1}{e^{hc/\lambda kT} - 1}\,. \tag{2}$$

[1] A 'black body' is an object whose surface absorbs all radiation incident on it.
[2] This discussion can also be found on Wikipedia.

For constant T (the absolute temperature), this equation describes a differentiable function of λ with a single maximum value. We find this value by the usual means:

$$\frac{\partial u_\lambda(\lambda, T)}{\partial \lambda} = 0\,, \tag{3}$$

yielding us

$$\frac{hc}{\lambda kT}\frac{e^{hc/\lambda kT}}{e^{hc/\lambda kT}-1} - 5 = 0\,. \tag{4}$$

Next, we lump our variable λ and constants together as such:

$$x = \frac{hc}{\lambda kT}\,, \tag{5}$$

and then (4) takes the simpler form

$$\frac{xe^x}{e^x-1} = 5\,, \tag{6}$$

The Mathematics

I intend to use the Lambert W function Lemma: Given

$$ze^z = B\,, \tag{7}$$

then, on taking the Lambert W function across this equation, we get

$$z = W(B)\,. \tag{8}$$

(See my notes on this, if you wish.)

3 The Solution

Going back to (1), we begin by clearing of fractions:

$$xe^x = 5(e^x - 1)\,, \tag{9}$$

With a little algebra, we can get

$$(x-5)e^x = -5\,. \tag{10}$$

On multiplying through by e^{-5}, we obtain

$$(x-5)e^{x-5} = -5e^{-5}\,. \tag{11}$$

At this point, we can apply the Lambert W function, to get

$$x - 5 = W_n(-5e^{-5})\,, \tag{12}$$

or

$$x = 5 + W_n(-5e^{-5})\,, \tag{13}$$

This expression $W_n(-5e^{-5})$ has at most two real values, one for case $n = 0$ and the other for case $n = -1$.

For case $n = -1$, we get $W_{-1}(-5e^{-5}) = -5$, which means that $x = 0$, the so-called trivial case, which we're not interested in.

For case $n = 0$, we get $W_0(-5e^{-5}) \approx -0.0348858$, which gives us the value for x

$$x = 5 + (-0.0348858) \approx 4.965\,. \tag{14}$$

4 Afterwards

When I took a modern physics course some decades ago, my textbook did not attempt to solve Eq. (1). Instead, it merely reported the results of the wavelength of maximum intensity. In effect, it skipped all of the solution presented here. But back then the Lambert W function was not well known to physicists,[3] and besides, they had approximation methods to solve equations like (1).

[3] This is probably still true, generally speaking.