Math Diversion Problem 515

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CONCERNING HORSES

Riding a horse is not a gentle hobby, to be picked
up and laid down like a game of
Solitaire. It is a grand passion.
— Ralph Waldo Emerson

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Title: A Lambert problem
 Presenter: Patrick
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1 The Problem

Given the relation

$$xe^{x^2} = y\,, \tag{1}$$

find the real values of x.

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{2}$$

then

$$z = W(B)\,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

The following is the 'Lambert W function base s'[1], or W_s, where s is a positive real number. Let's begin with the relation

$$xs^x = A\,, \tag{4}$$

[1] This notation I invented myself.

which looks very similar to (2). Then

$$x = W_s(xs^x) \equiv \frac{W(A\ln s)}{\ln s}\,. \tag{5}$$

But when $s = e$, we have that

$$x = W_e(xe^x) = \frac{W(A\ln e)}{\ln e} = W(A)\,, \tag{6}$$

which is the usual Lambert W function.

3 The Solution

The first thing I'll do is to square both sides of (1):

$$x^2\,(e^2)^{x^2} = y^2\,. \tag{7}$$

Now we're in the position to use the lemma in the previous section:

$$x^2 = W_{e^2}(y^2) = \frac{W(y^2\cdot\ln e^2)}{\ln e^2} = \frac{W(2y^2)}{2}\,, \tag{8}$$

or

$$x = \pm\frac{1}{\sqrt{2}}\sqrt{W(2y^2)}\,. \tag{9}$$