Math Diversion Problem 517

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> Zathras is used to being beast of burden to other people's needs. Very sad life... Probably have very sad death. But, at least there is symmetry.
> — Zathras (a character on Babylon 5)

The YouTube video is found at:

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Source: 3 years ago
Title: My Favorit Silly Identity (Members only)
 Presenter: BriTheMathGuy
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1 The Problem

Prove the relation

$$\sqrt{2}^{\sqrt{2}} = 2^{1/\sqrt{2}}\,. \tag{1}$$

2 The Solution

Method 0:

$$2^{1/\sqrt{2}} = 2^{\sqrt{2}/2} = \sqrt{2}^{\sqrt{2}}\,. \tag{2}$$

Method 1:

Let's take the logarithm base 2 across (1):

$$\sqrt{2}\log_2\sqrt{2} = \frac{1}{\sqrt{2}}\log_2 2 = \frac{1}{\sqrt{2}}\,. \tag{3}$$

We have to show that the LHS is equal to $1/\sqrt{2}$.

$$\sqrt{2}\log_2\sqrt{2} = \tfrac{1}{2}\sqrt{2}\log_2 2 = \tfrac{1}{2}\sqrt{2} = \frac{1}{\sqrt{2}}\,. \tag{4}$$

Done.

Method 2: (The 'overkill' method)

Let's take the natural logarithm across (1):

$$\sqrt{2}\ln\sqrt{2} = \frac{1}{\sqrt{2}}\ln 2\,. \tag{5}$$

I'll finish up by using the Lambert W function, though it can be finished without it. On taking the Lambert W function across (5), we get

$$\begin{aligned}\ln\sqrt{2} &= W(\frac{1}{\sqrt{2}}\ln 2)\\ &= W(\tfrac{1}{2}\sqrt{2}\ln 2)\\ &= W(\sqrt{2}\ln 2^{1/2})\\ &= W(\sqrt{2}\ln\sqrt{2})\\ &= \ln\sqrt{2}\,. \end{aligned}\tag{6}$$

And we've established it.