

Math Diversion Problem 518

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He that answereth a matter before he heareth it,
it is folly and shame unto him.
— Proverbs 18:13

This problem is found at:

Title: Is it irrational?
Presenter: Patrick

1 The Problem

Given that $\sqrt{2}$ is irrational,¹ is

$$\phi = \sqrt{3 + \sqrt{2}} \tag{1}$$

irrational?

Definition: A real number is said to be ‘rational’ if and only if it can be expressed as the ratio of two integers, the denominator not being zero.

2 The Solution

Lemma:

If ϕ^2 is irrational then ϕ is irrational.

Proof: Assume not. Assume that ϕ is rational. Then it can be expressed as

$$\phi = \frac{a}{b}, \tag{2}$$

where a, b are nonzero integers. Then

$$\phi^2 = \frac{a^2}{b^2}. \tag{3}$$

¹This is easy to show.

But both a^2 and b^2 are integers. Therefore, ϕ^2 has been expressed as the ratio of two integers, making it a rational number. Contradiction.

Then, from (1),

$$\phi^2 = 3 + \sqrt{2}. \quad (4)$$

So,

$$\phi^2 - 3 = \sqrt{2}. \quad (5)$$

But we were told the $\sqrt{2}$ is irrational (this is easy to prove). Therefore, $\phi^2 - 3$ is also irrational. Hence, ϕ^2 is irrational. But because of the above lemma, we know that

$$\phi = \sqrt{3 + \sqrt{2}}. \quad (6)$$

is irrational.