Math Diversion 519

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Don't ever take a fence down until you know
the reason it was put up.
— Chesterton

A typical induction problem.

1 The Problem

Use induction to prove this result.

Let $P(n)$ be the proposition that

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n} \tag{1}$$

for all integers $n \geq 2$.

2 The Preparation

Let m be the smallest integer on which the proposition is to be applied. (In this problem, $m = 2$, but in general it could be any integer.)

1) Show by brute calculation that $P(m)$ is true.
2) Assume $P(k)$ is true for some arbitrary $k \geq m$.
3) Show, on the basis of 1) and 2), that $P(k+1)$ is true.

Then, if one can successfully comply with all three of these requirements, then one can conclude that $P(n)$ is true for all n greater than or equal to the base case m.

Obviously we must proceed differently for constructing algorithms for finite products than for finite sums. To develop algebraic sentences equivalent to $P(k)$ and $P(k+1)$ for this class of problems we proceed as follows. Let $L(n)$ be the finite product

$$L(n) = f_m f_{m+1} \cdots f_n \,, \tag{2}$$

where $n \geq m$ and $L(n)$ is never zero. We note that

$$f_{k+1} = \frac{L(k+1)}{L(k)} \,. \tag{3}$$

Omitting the details, we assume that $L(k) = R(k)$, then $L(k+1) = R(k+1)$ if and only if

$$R(k+1) = R(k)\frac{L(k+1)}{L(k)} = R(k) f_{k+1} \,, \tag{4}$$

or

$$\frac{R(k+1)}{R(k)} = \frac{L(k+1)}{L(k)} = f_{k+1} \,. \tag{5}$$

We proceed under the understanding that it is impossible for $P(k)$ to be true for any $k \geq m$ (the inductive hypothesis) and for (5) to be true, but $P(k+1)$ to be false. In other words, by assuming $P(k)$, if we can show that (5) is true, then $P(k+1)$ is true, and that implies that $P(n)$ is established.

3 The Solution

Now to our given problem. Let $P(n)$ be the proposition that

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right)\cdots\left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n} \tag{6}$$

for all integers $n \geq 2$ (note: $m = 2$). Since $L(2) = 3/4 = R(2)$ then $P(2)$ is true. Now,

$$f_{k+1} = 1 - \frac{1}{(k+1)^2}$$

$$\begin{aligned} \frac{R(k+1)}{R(k)} &= \frac{(k+2)/2(k+1)}{(k+1)/2k} \\ &= \frac{k^2+2k}{(k+1)^2} \overset{\text{VE}}{=} \frac{(k^2+2k+1)-1}{(k+1)^2} \\ &= \frac{(k+1)^2-1}{(k+1)^2} \\ &= 1 - \frac{1}{(k+1)^2} \\ &= f_{k+1} \,. \end{aligned} \tag{7}$$

Therefore $P(n)$ is true.