# Math Diversion Problem 520

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First things first...But not necessarily in that order.
— Doctor Who

The problem is found at:

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Source: https://www.youtube.com/watch?v=Gh-pw1iMXiY
Title: Sweden Math Olympiad
 Presenter:  Math Booster
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## 1 The Problem

Determine the value of $x$ from the given information.

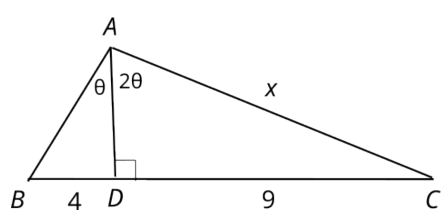


Figure 1. How would you go about to solve for $x$?

**Math Booster** solved this problem in a very geometric style. But I am much more familiar with algebra and trig than with Euclidean geometry. However, I will follow Math Booster's notion to add an auxiliary line to this problem, as follows:

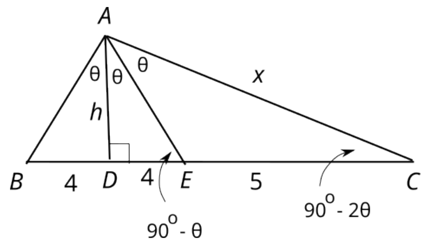


Figure 2. How would you go about to solve for $x$?

## 2 The Preparation

**The Law of Sines concerning triangles:**

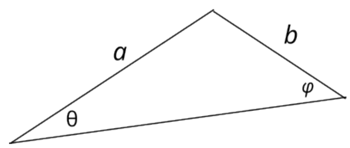


Figure 3. The Law of Sines requires two interior angles and their opposite sides.

The Law of Sines is the following rule that relates two interior angles of a triangle (which two doesn't matter) and their opposite sides, as follows:

$$\frac{\sin\varphi}{a} = \frac{\sin\theta}{b}\,, \tag{1a}$$

or

$$\frac{\sin\varphi}{\sin\theta} = \frac{a}{b}\,. \tag{1b}$$

A sine-cosine identity:

$$\sin\left(90^\circ - \theta\right) = \cos\theta\,. \tag{2}$$

A cotangent-tangent identity:

$$\cot 2\theta = \frac{1 - \tan^2\theta}{2\tan\theta}\,. \tag{3}$$

## 3 The Solution

The first relation to record is by the Pythagorean theorem:

$$x^2 = h^2 + 9^2\,. \tag{4}$$

Next, we apply the Law of Sines to triangle ADE:

$$\frac{\sin\theta}{\sin(90^\circ-\theta)} = \frac{4}{h}\,. \tag{5}$$

Now we use the sine-cosine identity:

$$\frac{\sin\theta}{\cos\theta} = \tan\theta = \frac{4}{h}\,. \tag{6}$$

Next, we apply the Law of Sines to triangle ACD:

$$\frac{\sin(90^\circ-2\theta)}{\sin 2\theta} = \frac{h}{9}\,. \tag{7}$$

Again, we use the sine-cosine identity:

$$\frac{\cos 2\theta}{\sin 2\theta} = \cot 2\theta = \frac{h}{9}\,. \tag{8}$$

Next, we apply the cotangent-tangent identity above, to get

$$\frac{1-\tan^2\theta}{2\tan\theta} = \frac{h}{9}\,. \tag{9}$$

After a bit of algebra, the becomes:

$$\tan^2\theta + \frac{2h}{9}\tan\theta - 1 = 0\,. \tag{10}$$

We can simplify this by using the fact that from (6), we can write $h\tan\theta = 4$; hence,

$$\tan^2\theta + \frac{8}{9} - 1 = 0\,. \tag{11}$$

Solving this, we have that, since $0 < \theta < 90$:

$$\tan\theta = \frac{1}{3}\,. \tag{12}$$

Plugging this back into (6), we get

$$h = 12\,. \tag{13}$$

Plugging this into (4), we get

$$x = 15\,. \tag{14}$$