Math Diversion 521

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Every good mathematician is at least half a philosopher,
and every good philosopher is at least
half a mathematician.
— Gottlob Frege

A typical problem in trigonometry with complex numbers.

1 The Problem

1) Use complex numbers to show that

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta\,, \tag{1a}$$

$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta\,. \tag{1b}$$

2) And then use these results to show that

$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\,, \tag{2}$$

2 The Preparation

$$e^{3ix} = \cos 3\theta + i\sin 3\theta\,, \tag{3}$$

and

$$e^{3ix} = (e^{ix})^3\,. \tag{4}$$

Expanding this last equation with the help of Pascal's triangle to expand the RHS: 1 3 3 1, we get

$$\text{Real part:} \qquad \cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta\,, \tag{5a}$$

$$\text{Imaginary part:} \qquad \sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta\,. \tag{5b}$$

3 The Solution

Let's start with the identity

$$\frac{\sin 3\theta}{\cos 3\theta} = \tan 3\theta\,. \tag{6}$$

On dividing (5b) by (5a), we have that

$$\tan 3\theta = \frac{3\cos^2\theta\sin\theta - \sin^3\theta}{\cos^3\theta - 3\cos\theta\sin^2\theta} \tag{7a}$$

$$= \frac{3\dfrac{\sin\theta}{\cos\theta} - \dfrac{\sin^3\theta}{\cos^3\theta}}{1 - 3\dfrac{\sin^2\theta}{\cos^2\theta}} \tag{7b}$$

$$= \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\,. \tag{7c}$$