Math Diversion Problem 525

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People often overlook the obvious.
— Doctor Who

The problem is found at:

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Source: https://www.youtube.com/watch?v=WwqW-fSj_jA
Title: The sum of a nilpotent and a unit is a unit
 Presenter:  Coconut Math
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1 The Problem

Let R be a commutative ring with unity element 1. If u is a unit in R and x is nilpotent in R, show that $x + u$ is also a unit in R.

Note: I am working this problem here (in an expanded form) because 1) it's such a cool bit of mathematics, and 2) because I can use it to introduce ring theory to my readers.

2 The Preparation: Part 1

A **ring** is a mathematical object where both an addition operation and a multiplication operation are defined. I will defer a more rigorous explanation of a ring to one of my monograph, which should be available soon. For the time being, you can think of the integers as one example of a ring, and the $n \times n$ matrices (for fixed $n > 1$) over the reals as another.

In a commutative ring the product of all elements commute, meaning that for all $a, b \in R$

$$ab = ba\,. \tag{1}$$

This product is also associative in every ring, commutative or not.

All rings have an additive identity element 0, but not all rings have a multiplicative identity element (the *unity*), usually represented as 1.

Definition: A ring R with unitary element 1 is said to be **a ring with unity** or a **unital ring**. From this point on, all rings will be considered as unital.

An element in a ring that does have an inverse (in the multiplicative sense) is called a **unit** in R. If u is such an element in R, then there exists some other element of R, say w, such that

$$wu = uw = \mathbf{1}\,. \tag{2}$$

Some people just write $w = u^{-1}$.

A nonzero element of a ring, x say, that has the property that

$$x^n = 0\,, \tag{3}$$

for some integer $n > 1$, is said to be **nilpotent**.

Let's take a break from the abstract to demonstrate an example each of a unit and a nilpotent element of a ring. For simplicity, let the ring in question be the set of 2×2 matrices with real entries, denoted as $M_2(\mathbb{R})$. Let

$$u = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\,. \tag{4}$$

If we define

$$w = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}\,, \tag{5}$$

we can show that $uw = \mathbf{1}$. I'll let the reader multiply these out mentally for combination uw:

$$uw = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\,, \tag{6}$$

and we recognize this rightmost matrix as the unity matrix for this ring. I leave it to the reader to show that $wu = \mathbf{1}$.

For an example of a nilpotent matrix x in R, let's examine

$$x = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\,. \tag{7}$$

I leave it to the reader to confirm that $x^2 = \mathbf{0}$, where

$$\mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\,, \tag{8}$$

where $\mathbf{0}$ is the additive inverse of this ring.

3 The Preparation: Part 2

Consider the factorization of the difference of two cubes,

$$x^3 - y^3 = (x - y)(x^2 + xy + x^2)\,, \tag{9a}$$

or of the sum of two cubes,

$$x^3 + y^3 = (x + y)(x^2 - xy + x^2)\,. \tag{9b}$$

How does this work that so many terms on the RHS reduce to just two terms on the LHS? Apparently there must be a lot of cancellation of terms during the expansion on the RHS. I leave it to the reader to prove this.

4 The Preparation: Part 3

Lemma: If $u \in R$ is a unit, then u^r is a unit in R for all $r > 1$.

Proof:

Since u is a unit, there exists some $w \in R$ such that

$$uw = 1\,. \tag{10}$$

Then u^r is also a unit because

$$u^r w^r = (uw)^r = 1\,. \tag{11}$$

The middle expression works because we've assumed that R is commutative.

5 The Solution

General factorization of the sum of two equal odd powers:

$$a^{2m+1} + b^{2m+1} = (a + b)(a^{2m} - a^{2m-1}b + a^{2m-2}b^2 - \cdots - ab^{2m-1} + b^{2m})\,. \tag{12}$$

The general factorization of the sum of two equal even powers:

$$a^{2m} + b^{2m} = (a + b)(a^{2m-1} - a^{2m-2}b + a^{2m-3}b^2 - \cdots - ab^{2m-2} + b^{2m-1})\,. \tag{13}$$

I'll show this for the odd power, but the proof is similar to the even power.

So, we are trying to show that $x + u$ is a unit. To do this, we have to show that there is some nonzero $y \in R$ such that

$$(x + u)y = 1\,. \tag{14}$$

Case Odd: The nilpotent element x conforms to the relation

$$x^n = 0\,, \tag{15}$$

for some integer $n \geq 2$. Set $2m+1 = n$ in (12), and also set $a = x$ and $b = u$. Then

$$x^n + u^n = (x+u)(x^{n-1} - x^{n-2}u + x^{n-3}u^2 - \cdots - xu^{n-2} + u^{n-1})\,, \tag{16}$$

which simplifies to

$$u^n = (x+u)(x^{n-1} - x^{n-2}u + x^{n-3}u^2 - \cdots - xu^{n-2} + u^{n-1})\,. \tag{17}$$

Next, we multiply through by w^n, to get

$$u^n w^n = (x+u)(x^{n-1} - x^{n-2}u + x^{n-3}u^2 - \cdots - xu^{n-2} + u^{n-1})w^n\,, \tag{18}$$

and use the above lemma (11) and simplify (and flip sides), to get

$$(x+u)[(x^{n-1} - x^{n-2}u + x^{n-3}u^2 - \cdots - xu^{n-2} + u^{n-1})w^n] = 1\,. \tag{19}$$

So, the factor y we are looking for in (14) is

$$y = (x^{n-1} - x^{n-2}u + x^{n-3}u^2 - \cdots - xu^{n-2} + u^{n-1})w^n\,. \tag{20}$$