Math Diversion Problem 528

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I love it when a plan comes together.
— Hannibal Smith, *The A-Team*

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=rLgZw5SNbXI
Title: Solving a 'Harvard' University entrance exam
 Presenter: The Map of Mathematics
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1 The Problem

Given the relation

$$x^{625} = 5^x \,, \tag{1}$$

find the values of x.

Note: I already solved this in Problem 426, where I used an alpha substitution. This time I'll try the Lambert W function.

By the way, WolframAlpha claims the solution to be

$$x = -\frac{625\,W\left(\frac{1}{625}(-1)^{1/625}\ln 5\right)}{\ln 5} \approx -0.997422 - 0.00500079i \tag{2}$$

Needless to say, this is not the answer I got last time.

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B \,, \tag{3}$$

then

$$z = W(B) \,, \tag{4}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

I also intend to use the following Lambert W function Lemma, that, for $a > 0$, given

$$za^z = B\,, \tag{5}$$

then

$$z = W_a(B)\,, \tag{6}$$

where

$$W_a(B) \equiv \frac{W(B \ln a)}{\ln a}\,, \tag{7}$$

which becomes the ordinary Lambert W function when $a = e$.

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3 The Solution (Note: $625 = 5^4$.)

The first thing I want to do is to take the 625th root across the Given equation, yielding

$$\beta^x = 1^{1/625} x\,, \tag{8}$$

where

$$\beta = 5^{1/625} \quad \text{(this is a real number)}\,, \tag{9}$$

and where $1^{1/625}$ is the complex number $e^{2\pi i/625}$.[1] Now, let's do a bit of algebra on (8) so that we can apply the Lambert Lemma discussed above.

$$1^{-1/625} = x\beta^{-x}\,. \tag{10}$$

Next, we multiply through by -1:

$$-1^{-1/625} = -x\beta^{-x}\,. \tag{11}$$

Now, we apply that lemma:

$$-x = W_\beta(-1^{-1/625})\,. \tag{12}$$

But,

$$-1^{-1/625} = e^{\pi i}(e^{-2\pi i/625}) = e^{-623\pi i/625} = e^{-\pi i/625} = (-1)^{1/625}\,. \tag{13}$$

On using this last result in (12) and solving for x, we have that

$$x = -W_\beta((-1)^{1/625}) = -\frac{W\big((-1)^{1/625}\ln(5^{1/625})\big)}{\ln(5^{1/625})} = -\frac{625\,W\big(\frac{1}{625}(-1)^{1/625}\ln 5\big)}{\ln 5}\,. \tag{14}$$

[1] Confession: I do not like having to extract complex roots from complicated problems like this one, because I'm not very good at it. Oh, well. Sigh!