Math Diversion 529

P. Reany

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You don't understand anything until you
learn it more than one way.
— Marvin Minsky

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=wxq60oaPjc0
Title: An Exponential Equation | Too Radical?
 Presenter: SyberMath
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1 The Problem

Given the relation

$$(2+\sqrt{3})^x = \sqrt[4]{2-\sqrt{3}}\,, \tag{1}$$

find the complex values of x.

2 The Solution

We can simplify (1) bu taking both sides to the fourth power:

$$(2+\sqrt{3})^{4x} = 2-\sqrt{3}\,. \tag{2}$$

Next, we multiply both sides by $2+\sqrt{3}$:

$$(2+\sqrt{3})^{4x+1} = (2-\sqrt{3})(2+\sqrt{3}) = 1\,. \tag{3}$$

Since we're looking for complex roots, we remember that

$$1 = e^{2i\pi n} \quad \text{for } n \in \mathbb{Z}\,. \tag{4}$$

On substituting this value into (3) and then taking the natural logarithm across the equation, we have that

$$4x+1 = \frac{2i\pi n}{\ln(2+\sqrt{3})}\,. \tag{5}$$

After some algebra,

$$x = \frac{1}{4}\Big[\frac{2i\pi n}{\ln(2+\sqrt{3})} - 1\Big] = \frac{1}{4}\Big[\frac{2i\pi n - \ln(2+\sqrt{3})}{\ln(2+\sqrt{3})}\Big] = \frac{\ln(2-\sqrt{3}) + 2i\pi n}{4\ln(2+\sqrt{3})}\,. \quad (6)$$

The real solution requires us to set $n = 0$, which gives us

$$x = -\frac{1}{4}\,. \quad (7)$$