Math Diversion 531

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In the middle of difficulty lies opportunity.
— John Archibald Wheeler

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=Tu6W6QRV9pc
Title: China | Can you solve this?
 Presenter: MathMinds
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1 The Problem

Given the relation

$$2^a = 3^b = 5^c \,, \tag{1}$$

find the values of

$$\phi = \frac{c}{a} + \frac{c}{b} \,. \tag{2}$$

And for fun, find the values of a, b, c.

2 The Solution

Apply to (1) log base 10.

$$a \log 2 = b \log 3 = c \log 5 \,, \tag{3}$$

From this last equation, we can extract ratios:

$$\frac{c}{a} = \frac{\log 2}{\log 5} \,, \qquad \frac{c}{b} = \frac{\log 3}{\log 5} \,. \tag{4}$$

Therefore,

$$\phi = \frac{\log 2}{\log 5} + \frac{\log 3}{\log 5} = \frac{\log 6}{\log 5} \,. \tag{5}$$

Now for the values of a, b, c. Let's begin with a change of variables:

$$2^{\alpha \log 3} = 3^{\beta \log 2} = 5^{\gamma \log 3} \,. \tag{6}$$

Again, we take the log base 10.

$$\alpha \log 3 \log 2 = \beta \log 2 \log 3 = \gamma \log 3 \log 5\,. \tag{7}$$

From the leftmost expression and the center, we get

$$\alpha = \beta = 1\,, \tag{8}$$

implying that

$$a = \log 3\,, \quad b = \log 2\,, \tag{9}$$

But we also have that

$$\beta = \gamma \frac{\log 5}{\log 2}\,, \tag{10}$$

hence,

$$\gamma = \frac{\log 2}{\log 5}\,. \tag{11}$$

Therefore,

$$c = \gamma \log 3 = \frac{\log 2\ \log 3}{\log 5}\,. \tag{12}$$

And finally,

$$\begin{aligned} \phi = \frac{c}{a} + \frac{c}{b} &= \frac{\log 2\ \log 3 \,/\, \log 5}{\log 3} + \frac{\log 2\ \log 3 \,/\, \log 5}{\log 2} && \text{(13a)} \\ &= \frac{\log 2}{\log 5} + \frac{\log 3}{\log 5} && \text{(13b)} \\ &= \frac{\log 2 + \log 3}{\log 5} && \text{(13c)} \\ &= \frac{\log 6}{\log 5}\,. && \text{(13d)} \end{aligned}$$

Comment: The lack of symmetry of these values for a, b, c indicate that these values are not unique, and that they can only be determined up to proportions.