Math Diversion Problem 532

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> You either get control of your lusts and feelings of entitlement, or they will get control of you.
> — The Author

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=vtTU3JBCpns
Title: A Nice Algebra Problem
 Presenter: SALogic
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1 The Problem

Given the relation

$$e^{x^2-1} = x\,, \tag{1}$$

find the values of x.

By the way, WolframAlpha claims the solution to be

$$x = 1\,, \qquad x = 0.450764 + 0i\,, \tag{2}$$

where $x = 1$ is the trivial solution.

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{3}$$

then

$$z = W(B)\,, \tag{4}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

I also intend to use the following Lambert W function Lemma, that, for $a > 0$, given

$$za^z = B\,, \tag{5}$$

then

$$z = W_a(B)\,, \tag{6}$$

where

$$W_a(B) \equiv \frac{W(B\ln a)}{\ln a}\,, \tag{7}$$

which becomes the ordinary Lambert W function when $a = e$.

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3 The Solution

Let's begin by squaring both sides:

$$(e^2)^{x^2-1} = x^2\,, \tag{8}$$

Next, we multiply through by $(e^2)^{-x^2}$:

$$(e^2)^{-1} = x^2(e^2)^{-x^2}\,. \tag{9}$$

On switching sides and multiplying through by -1, we have that

$$-x^2(e^2)^{-x^2} = -e^{-2}\,. \tag{10}$$

Now, we can apply the Lambert Lemma discussed above.

$$-x^2 = W_{e^2}(-e^{-2}) = \frac{W_n(-e^{-2}\ln e^2)}{\ln e^2} = \frac{W_n(-2e^{-2})}{2}\,. \tag{11}$$

Therefore,

$$x = \pm\, i\sqrt{\frac{W_n(-2e^{-2})}{2}}\,. \tag{12}$$

WolframAlpha gives for $W_0(-2e^{-2})$:

$$W_0(-2e^{-2}) = -0.40676\,. \tag{13}$$

Hence,

$$x_0 = \pm\, i\sqrt{\frac{W_n(-2e^{-2})}{2}} = \pm\, i(0.45098i) = \pm 0.45098\,, \tag{14}$$

of which the positive solution is close that given to us by WolframAlpha.

WolframAlpha gives for $W_{-1}(-2e^{-2})$:

$$W_{-1}(-2e^{-2}) = -2\,. \tag{15}$$

Clearly,

$$x_{-1} = \pm\, 1\,. \tag{16}$$

And just as clearly, only the positive solution works in (1).