Math Diversion 535

P. Reany

April 27, 2025

> Unless you try to do something beyond what you have
> already mastered, you will never grow.
> — Ralph Waldo Emerson

The YouTube video is found at:

```
Source: https://www.youtube.com/watch?v=-Hxaz4SkgSs
Title: International Math Competition Problem:
 Presenter: New Track Mathematics Video
```

1 The Problem

Given the relation

$$3^y + 3^{2y} = 3^{3y}\,, \tag{1}$$

find all values of y.

2 The Solution

Eq. (1) can first be rewritten as

$$3^{3y} - 3^{2y} - 3^y = 0\,, \tag{2}$$

and then as

$$3^y(3^{2y} - 3^y - 1) = 0\,, \tag{3}$$

Now, the function 3^y is never zero, even if y is complex. That leaves us to look for solutions to

$$3^{2y} - 3^y - 1 = 0\,. \tag{4}$$

But this is just a quadratic in 3^y:

$$(3^y)^2 - (3^y)^1 - 1 = 0\,. \tag{5}$$

Therefore,

$$3^y = \frac{1 \pm \sqrt{1 - 4(1)(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}\,. \tag{6}$$

If we are to find all 'all values', we must add in a factor of $e^{2\pi in}$:

$$3^y = \frac{1 \pm \sqrt{5}}{2}\, e^{2\pi in} \quad \text{where } n \in \mathbb{Z}\,. \tag{7}$$

Now we take the natural logarithm across this equation:

$$y \ln 3 = \ln \frac{1 \pm \sqrt{5}}{2} + 2\pi in \quad \text{where } n \in \mathbb{Z}\,. \tag{8}$$

And finally,

$$y = \frac{\ln \frac{1}{2}(1 \pm \sqrt{5}\,) + 2\pi in}{\ln 3} \quad \text{where } n \in \mathbb{Z}\,. \tag{9}$$

The real solution of this is

$$y = \frac{\ln \frac{1}{2}(1 + \sqrt{5}\,)}{\ln 3}\,. \tag{10}$$