Math Diversion Problem 539

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You have to know what to look for, so you can spot it.
— Papago Indian drug-enforcement
border scout

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=r35c9UBMxuI
Title: A Nice Algebra Problem
 Presenter: SALogic
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1 The Problem

Given the relation

$$(1/2)^x = x/8\,, \tag{1}$$

find the values of x.

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{2}$$

then

$$z = W(B)\,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

I also intend to use the following Lambert W function Lemma, that, for $a > 0$, given

$$za^z = B\,, \tag{4}$$

then

$$z = W_a(B)\,, \tag{5}$$

where

$$W_a(B) \equiv \frac{W(B \ln a)}{\ln a}\,, \tag{6}$$

which becomes the ordinary Lambert W function when $a = e$.

Another lemma I'll need from the theory of the Lambert W function is the following: If

$$y \ln y = B\,, \tag{7}$$

then

$$\ln y = W(y \ln y) = W(B)\,. \tag{8}$$

3 The Solution (Note: $256 = 2^8$.)

The trivial solution is $x = 2$, but are there any other solutions?

The first thing I want to do is to apply a little algebra to the Given equation, yielding

$$8 \cdot 2^{-x} = x\,, \tag{9}$$

and then to multiply across, to get

$$8 = x2^x\,. \tag{10}$$

Now, we can apply the first Lambert Lemma discussed above.

$$W_2(8) = x\,. \tag{11}$$

Continuing,

$$x = W_2(8) = \frac{W_n(8 \cdot \ln 2)}{\ln 2} = \frac{W_n(\ln 256)}{\ln 2}\,, \tag{12}$$

which is the general solution. We have two chances to get a real solution out of this, being either for $n = 0$ or for $n = -1$. WolframAlpha tells me that the $n = -1$ solution is nonreal, so that leaves the case $n = 0$.

$$x = \frac{W_0(\ln 256)}{\ln 2} = \frac{W_0(4 \ln 4)}{\ln 2} = \frac{\ln 4}{\ln 2} = \frac{2 \ln 2}{\ln 2} = 2\,, \tag{13}$$

where we used the other lemma above and that $W_0 = W$.