Math Diversion Problem 540

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April 28, 2025

In mathematics, you don't understand things.
You just get used to them.
— John von Neumann

The problem is found at:

`Source: A common problem in group theory`

1 The Problem

Let G be a group, and let H, K be subgroups of G. Show that $H \cap K$ (that is, the intersection of H and K) is a subgroup of G, expressed as:

$$H \cap K \leq G \,. \tag{1}$$

Note: We will express the group identity as e.

2 Solution

Since H and K are subgroups of G, their intersection cannot be empty, since both subgroups contain the group identity e, hence,

$$e \in H \cap K \,. \tag{2}$$

Now, whatever elements are in $H \cap K$ they're going to obey associativity of product because that feature is inherited from G. We have two more conditions to check: closure of products and closure of inverses.

Closure of products: If x, y are in $H \cap K$, is xy in $H \cap K$? The answer is yes, because for x, y in $H \cap K$ then x, y are in both H and K separately. Hence, xy is in both of them separately; hence, this product is in their intersection.

Closure of inverses: If x is in $H \cap K$, is x^{-1} in $H \cap K$? The answer is yes, because for x in $H \cap K$ then x is in both H and K separately. Hence, x^{-1} is in both of them separately; hence, this inverse is in their intersection.

Therefore, the intersection of two subgroups of a group is itself a subgroup of the group.