Math Diversion Problem 541

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> You can be creative in anything — in math, science, engineering, philosophy — as much as you can in music or in painting or in dance.
> — Ken Robinson

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=vLvURYicuLc
Title: An Interesting Equation With A Parameter
 Presenter: SyberMath
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1 The Problem

Given the relation

$$\sqrt{x+a} = e^x \quad (a > 0)\,, \tag{1}$$

find the values of x. (Note: I will approach the solution to this problem in the standard way, which is different than how SyberMath approached it.)

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{2}$$

then

$$z = W(B)\,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

I also intend to use the following Lambert W function Lemma, that, for $a > 0$, given

$$za^z = B\,, \tag{4}$$

then

$$z = W_a(B)\,, \tag{5}$$

where

$$W_a(B) \equiv \frac{W(B \ln a)}{\ln a}\,, \tag{6}$$

which becomes the ordinary Lambert W function when $a = e$.

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3 The Solution

The first thing I want to do is to square both sides of the Given equation, yielding

$$x + a = (e^2)^x\,, \tag{7}$$

and then make the variable change

$$y = x + a\,, \tag{8}$$

to give us

$$y = (e^2)^{y-a}\,. \tag{9}$$

After a bit of algebra, we get

$$-y(e^2)^{-y} = -(e^2)^{-a} = -e^{-2a}\,. \tag{10}$$

On applying W_{e^2} to this equation, we have that

$$-y = W_{e^2}(-e^{-2a}) = \frac{W_n(-e^{-2a} \ln e^2)}{\ln e^2} = \frac{W_n(-2e^{-2a})}{2}\,. \tag{11}$$

Converting back to variable x, gives us

$$x = -\frac{1}{2} W_n(-2e^{-2a}) - a\,, \tag{12}$$

which is close to what WolframAlpha got, which is

$$x = -\frac{1}{2} W(-2e^{-2a}) - a\,. \tag{13}$$