

Math Diversion Problem 542

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Solving math puzzles can enhance cognitive functions such
as problem-solving, logical reasoning, and memory,
contributing to overall brain health.

— Daniel Levitin
(renowned neuroscientist)

The YouTube video is found at:

Source: <https://www.youtube.com/watch?v=ZyH3FBzquQ0>

Title: YOU Must-Know ... to Prepare for International Math Olympiad

Presenter: Math-X

1 The Problem

Given the relation

$$x^{x^2} = 2, \tag{1}$$

find the real values of x .

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B, \tag{2}$$

then

$$z = W(B), \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

A lemma I'll need from the theory of the Lambert W function is the following:
If

$$y \ln y = B, \tag{4}$$

then

$$\ln y = W(y \ln y) = W(B). \quad (5)$$

3 The Solution

The Presenter did not use Lambert, but we will, and we will also extricate ourselves from it bilaterally. That means that it will disappear from the solution.

The first thing I want to do is to square both sides of the Given equation, yielding

$$(x^2)^{x^2} = 2^2, \quad (6)$$

and then take the natural logarithm, to get

$$(x^2) \ln(x^2) = 2 \ln 2. \quad (7)$$

Now, we can apply the Lambert Lemma discussed above.

$$\ln(x^2) = \ln 2. \quad (8)$$

Continuing, we have that

$$x^2 = 2, \quad (9)$$

which gives us the solutions

$$x = \pm\sqrt{2}. \quad (10)$$