

Structured Differentiation for Advanced Calculus

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Matters of notation play a considerable role in connection with the chain rule. Wide varieties of usage exist in mathematical writing where the chain rule is concerned.

—Taylor & Mann

Comment: I'm here republishing this article in smaller sections for this web format. Partial differentiation is a highly nontrivial subject. So let's try to make it make sense.

Introduction

The purpose of this paper is to present a structured, semantically unified formalism for differentiation to meet the needs of the undergraduate and graduate mathematics student. The problems in this paper are taken mostly from texts on advanced calculus. Formally, we shall refer to the formalism used herein as *structured differentiation* or SD. In particular, this article is a continuation of the article “A structured differentiation for physicists,” published in the *AJNP*¹ in January of 1992, which should be taken as a reference to the notation used in this article. The current article is an expansion of the article I published in the *AJNP* in April 1996.

Review of some important issues

Let's start by presenting a notational ambiguity presented by Buck ([?] p. 137–8). He refers to the ambiguity of partial derivatives. Let

$$w = f(x, u, v) \quad u = g(x, v, y) \quad v = h(x, y). \quad (1)$$

¹The *Arizona Journal of Natural Philosophy*.

For the derivative of w by x Buck offers:

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial x} + \frac{\partial w}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial w}{\partial v} \frac{\partial v}{\partial x} + \frac{\partial w}{\partial u} \frac{\partial u}{\partial v} \frac{\partial v}{\partial x} \quad (2)$$

Without further notation this equation is ambiguous. There are a number of solutions. One is to subscript $\frac{\partial w}{\partial x}$ on the right to get $\left(\frac{\partial w}{\partial x}\right)_y$, meaning that y is held constant during the differentiation. If we do this then we interpret $\frac{\partial w}{\partial x}$ on the left as the “total” partial derivative, and $\left(\frac{\partial w}{\partial x}\right)_y$ on the right as the “partial” partial derivative. More to the liking of authors Taylor & Mann ([2]) is to re-express (2) as

$$\frac{\partial w}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial u} \frac{\partial g}{\partial x} + \frac{\partial f}{\partial v} \frac{\partial h}{\partial x} + \frac{\partial f}{\partial u} \frac{\partial g}{\partial v} \frac{\partial h}{\partial x} \quad (3)$$

But this is still very unsatisfactory, so I think that Taylor & Mann would probably also introduce the primitive function $W(x, y) = f(x, v, y)$, and thus write

$$\frac{\partial W}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial u} \frac{\partial g}{\partial x} + \frac{\partial f}{\partial v} \frac{\partial h}{\partial x} + \frac{\partial f}{\partial u} \frac{\partial g}{\partial v} \frac{\partial h}{\partial x} \quad (4)$$

At least at this point we know exactly how to interpret $\frac{\partial W}{\partial x}$ because $W(x, y)$ is primitive in x . $\frac{\partial W}{\partial x}$ must be an explicit derivative and a total derivative. But this still leaves ambiguity in $\frac{\partial g}{\partial x}$ ($\frac{\partial f}{\partial u}$ and $\frac{\partial h}{\partial x}$ must be total/explicit derivatives). So, I think that we should also introduce the primitive function $G(x, y) = g(x, v, y)$, and so the equation becomes

$$\frac{\partial W}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial u} \frac{\partial G}{\partial x} + \frac{\partial f}{\partial v} \frac{\partial h}{\partial x} + \frac{\partial f}{\partial u} \frac{\partial g}{\partial v} \frac{\partial h}{\partial x} \quad (5)$$

At least by doing all these contortions we have removed the ambiguities. Now all the partial derivatives can be interpreted as explicit derivatives. And it should be remembered that the canonical form for displaying answers involving total derivatives is to represent them as explicit derivatives regardless of the formalism chosen.

But there is an easier way to do this derivative. Instead of overloading the partial derivative to be all things for all occasions, why not just define two other derivative symbols? Thus in SD we have the partial derivative meaning explicit derivative, the copartial derivative ∂ meaning implicit derivative, and the deltal derivative δ meaning the total derivative, and the three operators satisfy the equation

$$\frac{\delta}{\delta x} = \frac{\partial}{\partial x} + \frac{\partial}{\partial x} \quad (6)$$

which can also be written as

$$\delta_x = \partial_x + \partial_x \quad (7)$$

and which can be further simplified to

$$\delta = \partial + \not\partial. \quad (8)$$

With this complete differential operator we can operate on $w = w(x, u, v)$ to get

$$\frac{\delta w}{\delta x} = \frac{\partial w}{\partial x} + \frac{\not\partial w}{\not\partial x} \quad (9)$$

or,

$$\begin{aligned} \frac{\delta w}{\delta x} &= \frac{\partial w}{\partial x} + \frac{\partial w}{\partial u} \frac{\delta u}{\delta x} + \frac{\partial w}{\partial v} \frac{\delta v}{\delta x} \\ &= \frac{\partial w}{\partial x} + \frac{\partial w}{\partial u} \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial v} \frac{\delta v}{\delta x} \right) + \frac{\partial w}{\partial v} \frac{\delta v}{\delta x} \\ &= \frac{\partial w}{\partial x} + \frac{\partial w}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial w}{\partial u} \frac{\partial u}{\partial v} \frac{\delta v}{\delta x} + \frac{\partial w}{\partial v} \frac{\delta v}{\delta x}. \end{aligned} \quad (10)$$

Now, for the derivative of w by y , we get, given that $\partial w/\partial y \equiv 0$

$$\frac{\delta w}{\delta y} = \frac{\not\partial w}{\not\partial y} = \frac{\partial w}{\partial u} \frac{\delta u}{\delta y} + \frac{\partial w}{\partial v} \frac{\delta v}{\delta y} = \frac{\partial w}{\partial u} \left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial v} \frac{\delta v}{\delta y} \right) + \frac{\partial w}{\partial v} \frac{\delta v}{\delta y}. \quad (11)$$

As a final comment, the dotal derivative reduces to an ordinary derivative when the function being differentiated is a function of only one independent (fundamental) variable.

But Taylor & Mann have their problems trying to present a consistent notation and vocabulary too. On page 271 we find: Consider the function $G(x, y)$ as a function of u and y , with $x = f(u, y)$. The partial derivative with respect to y is

$$\frac{\partial G}{\partial x} \frac{\partial f}{\partial y} + \frac{\partial G}{\partial y} = \frac{\left| \frac{\partial(F, G)}{\partial(x, y)} \right|}{\frac{\partial F}{\partial x}} = 0, \quad (12)$$

where I have inserted the determinant symbols to conform to SD. As seen from the perspective of SD the phrase “partial derivative with respect to y ” is bizarre. First, because it doesn’t even bother to mention what is being differentiated! Of course it’s G that’s being differentiated, but the left-hand side of this equation is not a partial derivative, it’s a total derivative. In any case, the symbol $\partial G/\partial y$ is just a term on the left-hand side. In SD this confusion doesn’t occur, because for every descriptive meaning (such as “partial derivative”) there is a 1-1 correspondence between meaning, which itself has a 1-1 correspondence between itself and symbols. For example, in SD the phrase “partial derivative” has only one symbol, namely ∂ , and it has only one meaning, namely, it is an explicit derivative. It is probably safe to say that the only way to make sense of the standard notations/symbology of “partial differentiation” is to already know what it means. Even the term “partial differentiation” is a misnomer—it’s really “total differentiation.”

On page 175, problem 22 we find the problem “If $u = x^2 + y^2 + z^2$ and $z = xyv$, how many meanings are there for $\partial u/\partial y$?” Well, fortunately in SD it has only one!

Relation to the chain rule:

Before doing the solved problems, I want to show that the parametric split of the delta derivative into

$$\frac{\delta}{\delta x} = \frac{\partial}{\partial x} + \frac{\partial}{\partial x}$$

is consistent with the ordinary chain rule. Let $f = f(x, y, z(x, y))$ and let $\mathbf{x} = (x, y, z)^t$. Then, by the chain rule

$$\frac{\delta f}{\delta x} = \frac{\partial f}{\partial \mathbf{x}} \frac{\delta \mathbf{x}}{\delta x} = \frac{\partial f}{\partial x} \frac{\delta x}{\delta x} + \frac{\partial f}{\partial (y, z)} \frac{\delta (y, z)}{\delta x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial x} \quad (13)$$