Math Diversion 546

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> If others would think about mathematical truths as deeply
> and as continuously as I have, they
> would make my discoveries.
> — Carl Friedrich Gauss

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=Dm-iKG3hpo0
Title: A MIT Math Test Logarithm Problem
 Presenter: MyMathProgress
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1 The Problem

Given the relations

$$a \cdot 2^b = 8\,, \tag{1a}$$

$$a^b = 2\,, \tag{1b}$$

find the real values of

$$\phi = a^{\log_2 a}\; 2^{b^2}\,. \tag{2}$$

2 The Solution

The obvious place to start is to take (1a) to the bth power, to get,

$$a^b \cdot 2^{b^2} = 8^b = 2^{3b}\,, \tag{3}$$

which simplifies to

$$2 \cdot 2^{b^2} = 2^{3b}\,, \tag{4}$$

where we used (1b). This result further simplifies down to

$$2^{b^2} = 2^{3b-1}\,. \tag{5}$$

Applying the logarithm base 2 across this, we have that

$$b^2 - 3b + 1 = 0\,, \tag{6}$$

which has solutions

$$b_{\pm} = \frac{3 \pm \sqrt{5}}{2}\,. \tag{7}$$

From here on, I'll just consider the b_+ root, as the b_- root follows similarly.

Now that we know b_+, we can solve for a_+ from (1b):

$$a_+ = 2^{1/b_+} = 2^{\frac{1}{2}(3-\sqrt{5})}\,. \tag{8}$$

Note: These are the same values WolframAlpha got for both a_+ and b_+.

Now, to calculate ϕ_+, I think I'll start with $\log_2 \phi_+$.

$$\log_2 \phi_+ = (\log_2 a_+)^2 + b_+^2 \tag{9a}$$
$$= (\tfrac{1}{2}(3-\sqrt{5}))^2 + (\tfrac{1}{2}(3+\sqrt{5}))^2 \tag{9b}$$
$$= \tfrac{1}{4}(28) \tag{9c}$$
$$= 7\,. \tag{9d}$$

Hence,

$$\phi_+ = 2^7 = 128\,. \tag{10}$$