Structured Differentiation for Advanced Calculus

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> Matters of notation play a considerable role in connection with the chain rule. Wide varieties of usage exist in mathematical writing where the chain rule is concerned.
>
> —Taylor & Mann

Comment: I'm here republishing this article in smaller sections for this web format. Partial differentiation is a highly nontrivial subject. So let's try to make it make sense. Knowledge of how to multiply matrices together and how to take the multiplicative inverse of a 2×2 is necessary.

Introduction

The purpose of this paper is to present a structured, semantically unified formalism for differentiation to meet the needs of the undergraduate and graduate mathematics student. The problems in this paper are taken mostly from texts on advanced calculus.

Solved problems from Taylor & Mann

PROBLEM:

On page 190 problem 3 we find (in non-SD notation): If $z = f(x, y)$ is a solution of $F(x, y, z) = 0$ (with $F_{,3} \neq 0$), and if $H(x, y) = G(x, y, f(x, y))$, show that

$$\frac{\partial F}{\partial z}\frac{\partial H}{\partial y} = -\frac{\partial(F, G)}{\partial(y, z)}\,, \tag{1}$$

where $F_{,3}$ means that F is explicitly differentiated by z, the third variant in the variant list of F.

SOLUTION:

(This is one annoyance in SD: We have to specifically declare that the quantity in the right-hand-side of (1) is a determinant, though in standard advanced calculus this is unnecessary. The reason it needs to be included in SD is because the notation $\partial(\cdots)/\partial(\cdots)$ is in general an $m \times n$ matrix. I believe that the advantages of this notation far outweigh its disadvantages.)

First, we really don't need to introduce H, which is just the primitive rewrite of G. If we ignore H for now we can rewrite (1) in SD notation as

$$\frac{\partial F}{\partial z}\,\frac{\delta G}{\delta y} = -\left|\,\frac{\partial(F,G)}{\partial(y,z)}\,\right| . \tag{2}$$

Our starting point is the equation

$$F(x,y,z(x,y)) = F(\boldsymbol{\eta}, z(\boldsymbol{\eta})) = 0\,, \tag{3}$$

where $\boldsymbol{\eta} \equiv (x,y)^t$ and the raised t means transpose. We are purposely keeping the notation consistent with a matrix formulation since we will be going in and out of matrix representations as the situation calls. On differentiating (3) by $\boldsymbol{\eta}$ we get

$$\frac{\delta F(\boldsymbol{\eta}, z(\boldsymbol{\eta}))}{\delta \boldsymbol{\eta}} = \frac{\partial F}{\partial \boldsymbol{\eta}} + \frac{\partial F}{\partial z}\frac{\delta z}{\delta \boldsymbol{\eta}} = \frac{\partial F}{\partial \boldsymbol{\eta}} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial \boldsymbol{\eta}} = 0 \tag{4}$$

from which we get

$$\frac{\partial z}{\partial \boldsymbol{\eta}} = \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) = -{F_{,z}}^{-1}\left(\frac{\partial F}{\partial x}\,, \frac{\partial F}{\partial y}\right) . \tag{5}$$

Now

$$\frac{\delta G}{\delta \boldsymbol{\eta}} = \frac{\partial G}{\partial \boldsymbol{\eta}} + \frac{\partial G}{\partial z}\frac{\partial z}{\partial \boldsymbol{\eta}} = \frac{\partial G}{\partial \boldsymbol{\eta}} - {F_{,z}}^{-1}\,\frac{\partial G}{\partial z}\frac{\partial F}{\partial \boldsymbol{\eta}} . \tag{6}$$

Multiplying through by $F_{,z} = \partial F/\partial z$, we get

$$\frac{\partial F}{\partial z}\frac{\delta G}{\delta \boldsymbol{\eta}} = \left(\frac{\partial G}{\partial x}\frac{\partial F}{\partial z} - \frac{\partial F}{\partial x}\frac{\partial G}{\partial z}, \frac{\partial G}{\partial y}\frac{\partial F}{\partial z} - \frac{\partial F}{\partial y}\frac{\partial G}{\partial z}\right) = \left(-\frac{\partial(F,G)}{\partial(x,z)}, -\frac{\partial(F,G)}{\partial(y,z)}\right) . \tag{7}$$

And the desired result is the second component of the above.

Notice that when going from the compact notation of vector symbols to matrix formulation, we must be very careful to put the matrices in the correct order so that the multiplication is both well-defined and consistent.

LEMMA 1

Since the deltal derivative is a generalized total derivative, we can take the total derivative of a vector-valued function of a vector variable. Let's do that now for a specific form of functional dependence. Let the function $\mathbf{f}(\mathbf{u}, \mathbf{v}) = 0$, where $\mathbf{u}$ is the fundamental of $\mathbf{f}$ and $\mathbf{v} = \mathbf{v}(\mathbf{u})$; then

$$\frac{\delta \mathbf{f}}{\delta \mathbf{u}} = \frac{\partial \mathbf{f}}{\partial \mathbf{u}} + \frac{\partial \mathbf{f}}{\partial \mathbf{v}}\frac{\delta \mathbf{v}}{\delta \mathbf{u}} = 0\,. \tag{8}$$

Therefore, if $\partial\mathbf{f}/\partial\mathbf{v}$ is assumed to be an invertible square matrix, we get

$$\frac{\delta\mathbf{v}}{\delta\mathbf{u}} = -\left(\frac{\partial\mathbf{f}}{\partial\mathbf{v}}\right)^{-1}\frac{\partial\mathbf{f}}{\partial\mathbf{u}}\,. \tag{9}$$

If $\mathbf{v}$ happens to be primitive with respect to $\mathbf{u}$ then this last equation can be written as

$$\frac{\partial\mathbf{v}}{\partial\mathbf{u}} = -\left(\frac{\partial\mathbf{f}}{\partial\mathbf{v}}\right)^{-1}\frac{\partial\mathbf{f}}{\partial\mathbf{u}}\,. \tag{10}$$

PROBLEM:

From page 191 , problem 4 reads: $u = f(x,y,z)$, $v = g(x,y,z)$ are solutions of $F(x,y,z,u,v) = 0$, $G(x,y,z,u,v) = 0$. Let $K(x,y,z) = H(x,y,z,f(x,y,z),g(x,y,z))$. Show that (under suitable conditions)

$$\frac{\partial K}{\partial z} = \frac{\left|\dfrac{\partial(F,G,H)}{\partial(z,u,v)}\right|}{\left|\dfrac{\partial(F,G)}{\partial(u,v)}\right|}\,. \tag{11}$$

SOLUTION:

We know how to interpret $\partial K/\partial z$ because K is the primitive reduction of H — that makes the partial here an explicit derivative just as in SD, but K was only introduced so that the explicit derivative would also be a total derivative. But SD has a total derivative operator, so we will dispense with K and instead solve for $\delta H/\delta z$, since it is equal to $\partial K/\partial z$. First, we let $\mathbf{F} \equiv (F,G)^t$, $\mathbf{x} \equiv (x,y,z)^t$, and $\mathbf{u} \equiv \mathbf{u}(\mathbf{x})$, then our system becomes

$$\frac{\delta\mathbf{F}}{\delta\mathbf{x}} = \frac{\partial\mathbf{F}}{\partial\mathbf{x}} + \frac{\partial\mathbf{F}}{\partial\mathbf{u}}\frac{\partial\mathbf{u}}{\partial\mathbf{x}} = 0\,, \tag{12}$$

so from Lemma 1 we get

$$\frac{\partial\mathbf{u}}{\partial\mathbf{x}} = -\left(\frac{\partial\mathbf{F}}{\partial\mathbf{u}}\right)^{-1}\frac{\partial\mathbf{F}}{\partial\mathbf{x}}\,. \tag{13}$$

This result will be substituted in this next equation:

$$\frac{\delta H}{\delta\mathbf{x}} = \frac{\partial H}{\partial\mathbf{x}} + \frac{\partial H}{\partial\mathbf{u}}\frac{\partial\mathbf{u}}{\partial\mathbf{x}}\,, \tag{14}$$

to get

$$\begin{aligned}\frac{\delta H}{\delta\mathbf{x}} &= \frac{\partial H}{\partial\mathbf{x}} + \frac{\partial H}{\partial\mathbf{u}}\left[-\left(\frac{\partial\mathbf{F}}{\partial\mathbf{u}}\right)^{-1}\frac{\partial\mathbf{F}}{\partial\mathbf{x}}\right] \\ &= \left(\frac{\partial H}{\partial\mathbf{x}}\left|\frac{\partial\mathbf{F}}{\partial\mathbf{u}}\right| - \frac{\partial H}{\partial\mathbf{u}}\,\mathrm{adj}\left[\frac{\partial\mathbf{F}}{\partial\mathbf{u}}\right]\frac{\partial\mathbf{F}}{\partial\mathbf{x}}\right)\left|\frac{\partial\mathbf{F}}{\partial\mathbf{u}}\right|^{-1}\,. \end{aligned} \tag{15}$$

Upon expanding this and then focusing on only the z component, we get

$$\frac{\partial K}{\partial z} = \frac{\delta H}{\delta z} = \left(\left| \frac{\partial(F_1, F_2)}{\partial(u_1, u_2)} \right| \frac{\partial H}{\partial z} - \left| \frac{\partial(H, F_2)}{\partial(u_1, u_2)} \right| \frac{\partial F_1}{\partial z} + \left| \frac{\partial(H, F_1)}{\partial(u_1, u_2)} \right| \frac{\partial F_2}{\partial z} \right) \left| \frac{\partial \mathbf{F}}{\partial \mathbf{u}} \right|^{-1}$$

$$= \frac{\left| \dfrac{\partial(F, G, H)}{\partial(z, u, v)} \right|}{\left| \dfrac{\partial(F, G)}{\partial(u, v)} \right|}. \quad (16)$$