Math Diversion Problem 552

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> This is the normal result of any idea you have in
> mathematics: It turns out they're either wrong
> or trivial or a known result.
> — Richard Bocherds

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=kOPR-V2-ggc
Title: A nice algebra question
 Presenter: mathmasteryminds
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1 The Problem

Given the relation

$$4^{-x} = x\,, \tag{1}$$

find the real values of x.

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{2}$$

then

$$z = W(B)\,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

A lemma I'll need from the theory of the Lambert W function is the following: If

$$y \ln y = B\,, \tag{4}$$

then

$$\ln y = W(y \ln y) = W(B)\,. \tag{5}$$

The following is the 'Lambert W function base s'[1], or W_s, where s is a positive real number. Let's begin with the relation

$$xs^x = A\,, \tag{6}$$

which looks very similar to (2). Then

$$x = W_s(xs^x) \equiv \frac{W(A \ln s)}{\ln s}\,. \tag{7}$$

But when $s = e$, we have that

$$x = W_e(xe^x) = \frac{W(A \ln e)}{\ln e} = W(A)\,, \tag{8}$$

which is the usual Lambert W function. (By the way, the proof to this lemma is not hard. It begins with setting $s^x = e^y$ and proceeding from there.)

If s is an integer, I may resort to putting parentheses around it to distinguish it from the n-series, as such $W_{(s)}$.

3 The Solution

Although this problem can be solved by inspection, we use Lambert for the practice.

So, using a little algebra on the Given equation, we get

$$x4^x = 1\,, \tag{9}$$

and then take the $W_{(4)}$ operator on it,[2] to get

$$x = W_{(4)}(1) = \frac{W_n(1 \cdot \ln 4)}{\ln 4} = \frac{W_n(2 \ln 2)}{2 \ln 2} \quad n \in \mathbb{Z}\,. \tag{10}$$

Now, we can apply the other Lambert Lemma discussed above applied to the principal value:

$$x_0 = \frac{W_0(2 \ln 2)}{2 \ln 2} = \frac{W_0(2 \ln 2)}{2 \ln 2} = \frac{\ln 2}{2 \ln 2} = \frac{1}{2}\,. \tag{11}$$

[1] This notation I invented myself.

[2] I put the integer 4 in parentheses so it won't be confused with an integer in the n-series.