Math Diversion Problem 555

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People are much more interested in learning mathematics
when they discover they need it for something.
— Richard Bocherds

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=VKp2j9qD8ts
Title: United kingdom l how to solve this nice math Olympiad problem
 Presenter: J Educational Tutorials
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1 The Problem

Given the relation

$$6^x = 6x + 24\,, \tag{1}$$

find all values of x and specify the two real values of x.

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{2}$$

then

$$z = W(B)\,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

A lemma I'll need from the theory of the Lambert W function is the following: If

$$y \ln y = B\,, \tag{4}$$

then

$$\ln y = W(y \ln y) = W(B)\,. \tag{5}$$

The following is the 'Lambert W function base s'[1], or W_s, where s is a positive real number. Let's begin with the relation

$$xs^x = A\,, \tag{6}$$

which looks very similar to (2). Then

$$x = W_s(xs^x) \equiv \frac{W(A\ln s)}{\ln s}\,. \tag{7}$$

But when $s = e$, we have that

$$x = W_e(xe^x) = \frac{W(A\ln e)}{\ln e} = W(A)\,, \tag{8}$$

which is the usual Lambert W function. (By the way, the proof to this lemma is not hard. It begins with setting $s^x = e^y$ and proceeding from there.)

If s is an integer, I may resort to putting parentheses around it to distinguish it from the n-series, as such $W_{(s)}$.

3 The Solution

First, let's divide the Given relation through by 6:

$$6^{x-1} = x + 4\,, \tag{9}$$

Next, we set a change of variables to

$$y = x + 4\,, \quad x - 1 = y - 5\,, \tag{10}$$

yielding,

$$6^{y-5} = y\,. \tag{11}$$

With some more algebra, we get:

$$-y6^{-y} = -6^{-5}\,. \tag{12}$$

Using one of the lemmas above, we have that

$$-y = W_{(6)}(-6^{-5}) = \frac{W_n(-6^{-5}\cdot\ln 6)}{\ln 6} = \frac{W_n(6^{-5}\cdot\ln 6^{-1})}{\ln 6} \quad n \in \mathbb{Z}\,. \tag{13}$$

Going back to x, we get the general solutions:

$$x_n = -\frac{W_n(6^{-5}\cdot\ln 6^{-1})}{\ln 6} - 4 \quad n \in \mathbb{Z}\,. \tag{14}$$

[1] This notation I invented myself.

From this we can derive two real solutions, starting with the $n = 0$ solution, otherwise known as the principal value based solution. According to a Lambert Lemma discussed above, we get:

$$\begin{aligned}
x_0 &= -\frac{W_0(6^{-5} \cdot \ln 6^{-1})}{\ln 6} - 4 && \text{(15a)} \\
&= -\frac{W_0([6 \cdot 6^{-1}]\ 6^{-5} \cdot \ln 6^{-1})}{\ln 6} - 4 && \text{(15b)} \\
&= -\frac{W_0(6^{-6} \cdot \ln 6^{-6})}{\ln 6} - 4 && \text{(15c)} \\
&= -\frac{\ln 6^{-6}}{\ln 6} - 4 && \text{(15d)} \\
&= \frac{6 \ln 6}{\ln 6} - 4 && \text{(15e)} \\
&= 2\,. && \text{(15f)}
\end{aligned}$$

This last result we could have figured out by inspection on the Given relation.

Our last possible real solution comes from the x_{-1} equation, which I'll employ WolframAlpha to assist me to get.

$$\begin{aligned}
x_{-1} &= -\frac{W_{-1}(6^{-5} \cdot \ln 6^{-1})}{\ln 6} - 4 && \text{(16a)} \\
&= -\frac{W_{-1}\left(-\frac{\ln 6}{7776}\right)}{\ln 6} - 4 && \text{(16b)} \\
&\approx \frac{10.7506}{1.79176} - 4 && \text{(16c)} \\
&\approx 6.000024 - 4 && \text{(16d)} \\
&\approx 2.000024\,. && \text{(16e)}
\end{aligned}$$

By the way, the command I used in WoframAlpha to calculate $W_{-1}(6^{-5} \cdot \ln 6^{-1})$ was

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ProductLog[-1, - 6^{-5}\ln (6)]//N
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ProductLog is the Mathematica command for the Lambert W function, and the '//N' command tells it to translate the result to a numerical approximation.

4 Conclusion

Apparently I've made a mistake somewhere, but I have spent enough time tracking it down without success. WolframAlpha tells me that the original equation has the two real solutions:

$$x = 2\,, \quad -3.99987\,. \tag{17}$$