Math Diversion 558

P. Reany

May 4, 2025

1 Solutions

On page 63 of NFCM [1], we find Problem (3.8):

Let $\mathbf{B} = \frac{1}{2}B_{\ell p}\,\boldsymbol{\sigma}_\ell \wedge \boldsymbol{\sigma}_p$ (sum on repeated indices) be a bivector and $\mathbf{b} = b_k\boldsymbol{\sigma}_k$ (sum on repeated indices) be a vector and they are related by the equation

$$\mathbf{B} = i\mathbf{b}\,. \tag{1}$$

Prove that $B_{ij} = \epsilon_{ijk}b_k$, where $\epsilon_{ijk} \equiv i^\dagger\boldsymbol{\sigma}_i \wedge \boldsymbol{\sigma}_j \wedge \boldsymbol{\sigma}_k$.

Lemma:

$$\boldsymbol{\sigma}_\ell \wedge \boldsymbol{\sigma}_p \cdot \boldsymbol{\sigma}_i \wedge \boldsymbol{\sigma}_j = \delta_{pi}\delta_{\ell j} - \delta_{pj}\delta_{i\ell}\,. \tag{2}$$

This is left to the reader to prove.

Proof of main result:
Expanding (1), we have

$$\tfrac{1}{2}B_{\ell p}\,\boldsymbol{\sigma}_\ell \wedge \boldsymbol{\sigma}_p = ib_k\boldsymbol{\sigma}_k\,. \tag{3}$$

On dotting through by $\boldsymbol{\sigma}_i \wedge \boldsymbol{\sigma}_j$ on the right and using (2), we have

$$\tfrac{1}{2}B_{\ell p}\,(\delta_{pi}\delta_{\ell j} - \delta_{pj}\delta_{i\ell}) = b_k(i\boldsymbol{\sigma}_k)\cdot(\boldsymbol{\sigma}_i \wedge \boldsymbol{\sigma}_j) = b_k\langle i\boldsymbol{\sigma}_k(\boldsymbol{\sigma}_i \wedge \boldsymbol{\sigma}_j)\rangle\,. \tag{4}$$

Which becomes

$$\tfrac{1}{2}(B_{ji} - B_{ij}) = b_k(i\boldsymbol{\sigma}_k)\cdot(\boldsymbol{\sigma}_i \wedge \boldsymbol{\sigma}_j) = b_k\langle i\boldsymbol{\sigma}_k(\boldsymbol{\sigma}_i \wedge \boldsymbol{\sigma}_j)\rangle\,. \tag{5}$$

And this becomes

$$-B_{ij} = -b_k(i^\dagger\boldsymbol{\sigma}_i \wedge \boldsymbol{\sigma}_j \wedge \boldsymbol{\sigma}_k)\,. \tag{6}$$

And finally, we get that

$$B_{ij} = \epsilon_{ijk}b_k\,. \tag{7}$$

References

[1] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.