Math Diversion 562

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Small moves, Ellie. Small moves.
— from the movie, *Contact*

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=ursa591tldU
Title: An Interesting Trigonometric Equation
 Presenter: SyberMath
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1 The Problem

Given the relation

$$\tan x \ \tan(x+1) = 1\,, \tag{1}$$

solve for x.

2 The Preparation

$$\tan(u \pm v) = \frac{\tan u \pm \tan v}{1 \mp \tan u \tan v} \tag{2}$$

3 The Solution

Let's start with a variable substitution:

$$x = y - 1/2\,, \quad x + 1 = y + 1/2\,. \tag{3}$$

Then, (1) becomes

$$\tan(y - 1/2) \ \tan(y + 1/2) = 1\,. \tag{4}$$

Next, we use the tangent identity in (2):

$$\frac{\tan y + \tan \frac{1}{2}}{1 - \tan y \tan \frac{1}{2}} \ \frac{\tan y - \tan \frac{1}{2}}{1 + \tan y \tan \frac{1}{2}} = 1\,. \tag{5}$$

On multiplying, we have that

$$\frac{\tan^2 y - \tan^2 \frac{1}{2}}{1 - \tan^2 y \tan^2 \frac{1}{2}} = 1\,. \tag{6}$$

On cross multiplying and collecting on terms, we get

$$\tan^2 y - \tan^2 \tfrac{1}{2} = 1 - \tan^2 y \tan^2 \tfrac{1}{2}\,, \tag{7}$$

then

$$(1 + \tan^2 \tfrac{1}{2}) \tan^2 y = (1 + \tan^2 \tfrac{1}{2})\,. \tag{8}$$

Now we cancel factors, leaving us with

$$\tan^2 y = 1\,. \tag{9}$$

On taking the square root, we get

$$\tan y = \pm 1\,. \tag{10}$$

So, the solutions in y are then

$$y_{\pm} = \pm\frac{\pi}{4} + n\pi = \pi\big(n \pm \frac{1}{4}\big)\,. \tag{11}$$

Therefore,

$$x_{\pm} = \pm\frac{\pi}{4} + n\pi = \pi\big(n \pm \frac{1}{4}\big) - \frac{1}{2}\,. \tag{12}$$

Well, my answer is close to what Sybermath got, but not too close to what WolframAlpha got.