Math Diversion Problem 563

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> Keep an open mind. That's the secret.
> — Doctor Who

> There are three rules for writing. Unfortunately,
> nobody knows what they are.
> — Somerset Maugham

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=AAn_ZTsExiM
Title: A very Difficult Logarithm & Exponents Question
 Presenter: Maths & Olympiad
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1 The Problem

Given the relation

$$5^{\log_2 \sqrt[3]{x}} \cdot \sqrt{x}^{\log_{2\sqrt{2}} 3} = 15^2 \,, \tag{1}$$

find the real values of x.

2 The Solution

The plan: Make those variable substitutions that simplify the math expressions.

First variable substitution: Let

$$x = y^6 \,. \tag{2}$$

Then (1) becomes

$$5^{\log_2 y^2} \cdot y^{3 \log_{2\sqrt{2}} 3} = 15^2 \,, \tag{3}$$

Second variable substitution: Let

$$y = 2^\alpha \,. \tag{4}$$

Then (3) becomes

$$5^{2\alpha} \cdot 2^{3\alpha \log_{2\sqrt{2}} 3} = 15^2 \,. \tag{5}$$

Take both sides to the 3/2 power:

$$5^{3\alpha} \cdot 2^{\frac{3}{2}\alpha \log_{2\sqrt{2}} 3} = 15^3 \,, \tag{6}$$

which becomes

$$5^{3\alpha} \cdot (2^{\frac{3}{2}})^{\log_{2\sqrt{2}} 3^\alpha} = 15^3 \,. \tag{7}$$

But this simplifies to

$$5^{3\alpha} \cdot 3^{3\alpha} = 15^3 \,, \tag{8}$$

or, by inspection, we have that

$$\alpha = 1 \,. \tag{9}$$

Thus, from (4), we get that

$$y = 2 \,. \tag{10}$$

Also, from (2), we get that

$$x = 2^6 = 64 \,. \tag{11}$$