Math Diversion 565

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The man who does not read has no advantage
over the man who cannot read.
— Mark Twain

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Source: The Ether of Mathematical Ideas
Title: How to eliminate the first-order term
 Presenter: Patrick
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1 The Problem

Given the relation

$$\frac{1}{x^2} + \frac{1}{(x+1)^2} = 1\,, \tag{1}$$

find the complex values of x.

2 The Solution

First, let's clear (1) of fractions.

$$(x+1)^2 + x^2 = x^2(x+1)^2\,. \tag{2}$$

Next, make the variable substitutions:

$$x = y - \tfrac{1}{2}\,, \quad x + 1 = y + \tfrac{1}{2}\,. \tag{3}$$

Then (2) becomes

$$(y + \tfrac{1}{2})^2 + (y - \tfrac{1}{2})^2 = (y - \tfrac{1}{2})^2(y + \tfrac{1}{2})^2\,. \tag{4}$$

Expanding and simplifying this, we have that

$$y^4 - \frac{5}{2}y^2 - \frac{7}{16} = 0\,. \tag{5}$$

We can solve this for y^2:[1]

$$y^2 = \frac{5}{2} \pm \sqrt{2}\,. \tag{6}$$

[1] So far, WolframAlpha agrees with me.

Now, I'll let WolframAlpha do the heavy lifting for me.

$$y_c = \pm\frac{i}{2}\sqrt{4\sqrt{2} - 5}\,, \tag{7a}$$

$$y_r = \pm\frac{1}{2}\sqrt{4\sqrt{2} + 5}\,. \tag{7b}$$

Hence,

$$x_c = \pm\frac{i}{2}\sqrt{4\sqrt{2} - 5} - \frac{1}{2}\,, \tag{8a}$$

$$x_r = \pm\frac{1}{2}\sqrt{4\sqrt{2} + 5} - \frac{1}{2}\,. \tag{8b}$$