Math Diversion Problem 566

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I love it when a plan comes together. — Hannibal Smith

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=Bm88kuDqcpk
Title: Harvard University Interview Math Tips & Tricks
 Presenter: Smart Math Tricks
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1 The Problem

Given the relation

$$x^y - y^x = 17\,, \tag{1}$$

find the integer values of x, y.

2 The Solution

I want to solve this problem by exploiting the symmetry in the table below.

$y \backslash x$	1	2	3	4	5	6
1	$\boxed{1}$	2	3	4	5	6
2	1	$\boxed{4}$	9	16	25	36
3	1	8	$\boxed{27}$	64	125	216
4	1	16	81	$\boxed{256}$	625	1296
5	1	32	243	1024	$\boxed{3125}$	7776
6	1	64	729	4096	15625	$\boxed{46656}$

Table 1. The y values take-up the leftmost column. The x values run across the top. The table entries are x^y. The main diagonal of the matrix is denoted by the boxed entries.

Now, if we imagine anti-diagonal lines going through the table, through the entries in the manner of (1,2), (8,9), (81,64), (1024, 625), (15625, 7776), and a bunch more that I didn't indicate, then we can notice that the entries equidistant from the main diagonal are in relation to each other as x^y to y^x. In other words, x and y effectively swapped places with each other.

But according to (1), this is convenient: All we have to do then is to systematically go through the table, subtracting the entries on the anti-diagonals, top-right from bottom-left. So let's do so.

$$1 - 2 = -1 \quad 8 - 9 = -1 \quad 81 - 64 = 17\checkmark\,. \tag{2}$$

Note: Read off the x, y values from the bottom-left triangle of the table.

So, going back to the table, we find that entry $81 = 3^4$, therefore,

$$x = 3 \quad y = 4\,. \tag{3}$$

3 Afterword

Now that the problem has been solved, as I look back at the table, it seems clear that I needn't have bothered to include in it row 6 and column 6.

Nevertheless, however one is to attack this problem, it must be done systematically. After all, what if instead of (1), we had been given the equation

$$x^y - y^x = -7849\,, \tag{4}$$

or something even worse?

So, the question is: How should a problem-solver balance trying a questionable shortcut method to what appears to be a special problem, against using general methods that would work efficiently against a wide variety of similar problems? For myself, if I don't see a 'shortcut' method quickly, I'll probably opt for more general techniques, because that way I get experience solving a wider class of problems. In other words, it's more efficient in the long run.