Math Diversion Problem 567

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Self-education is, I firmly believe, the only
kind of education there is.
— Isaac Asimov

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=tblj-TPkOfU
Title:A tricky question from old math book (1955 year)
 Presenter: Higher Mathematics
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1 The Problem

Given the relation

$$x^x = 2^{2048}\,, \tag{1}$$

find all real values of x.

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{2}$$

then

$$z = W(B)\,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

A lemma I'll need from the theory of the Lambert W function is the following: If

$$y \ln y = B\,, \tag{4}$$

then

$$\ln y = W(y \ln y) = W(B)\,. \tag{5}$$

The following is the 'Lambert W function base s'[1], or W_s, where s is a positive real number. Let's begin with the relation

$$xs^x = A\,, \tag{6}$$

which looks very similar to (2). Then

$$x = W_s(xs^x) \equiv \frac{W(A\ln s)}{\ln s}\,. \tag{7}$$

But when $s = e$, we have that

$$x = W_e(xe^x) = \frac{W(A\ln e)}{\ln e} = W(A)\,, \tag{8}$$

which is the usual Lambert W function. (By the way, the proof to this lemma is not hard. It begins with setting $s^x = e^y$ and proceeding from there.)

If s is an integer, I may resort to putting parentheses around it to distinguish it from the n-series, as such $W_{(s)}$.

3 The Solution

(Note: $2048 = 2^{11}$.)

Method 1: α substitution

Let

$$x = 2^\alpha\,. \tag{9}$$

Then the Given becomes

$$2^{\alpha\cdot 2^\alpha} = 2^{2^{11}}\,. \tag{10}$$

On equating exponents, we get

$$\alpha \cdot 2^\alpha = 2^{11}\,. \tag{11}$$

Now let[2]

$$\alpha = 2^\beta\,. \tag{12}$$

Then (11) becomes

$$2^\beta \cdot 2^{2^\beta} = 2^{11}\,. \tag{13}$$

On equating exponents, we have that

$$\beta + 2^\beta = 11\,. \tag{14}$$

So, we need a small odd integer to solve this, which is $\beta = 3$, which implies that $\alpha = 8$, which then implies that

$$x = 2^8 = 256\,. \tag{15}$$

[1] This notation I invented myself.

[2] Or solve for α in your head.

Method 2: Lambert W

Let's take the natural logarithm across (1), to get:

$$x \ln x = 2048 \ln 2 = 2^{11} \ln 2 = 2^3 2^8 \ln 2 = 8 \cdot 2^8 \ln 2 = 2^8 \ln 2^8 \,. \tag{16}$$

Now we take the Lambert W function across this equation, yielding,

$$\ln x = W(2^8 \ln 2^8) = \ln 2^8 \,. \tag{17}$$

This gives us

$$x = 2^8 = 256 \,. \tag{18}$$

When I presented the Given to WolfamAlpha, it returned to me

$$x = e^{W(2028 \ln(2))} \approx 253.881 \tag{19}$$

for the real solution for x. So, I suppose it does even bother to try (5) to reduce forms like

$$W(2028 \ln 2) \,. \tag{20}$$