Math Diversion Problem 568

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This is the normal result of any idea you have in
mathematics: It turns out they're either wrong
or trivial or a known result.
— Richard Bocherds

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=8825HXv3ADs
Title: Logarithm - System of Equations
 Presenter: Maths & Olympiad
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1 The Problem

Given the relations

$$\log_x y + \log_y x = \frac{26}{5}\,, \tag{1a}$$

$$xy = 64\,, \tag{1b}$$

find the values of x, y.

2 The Solution

On noting that $(\log_x y)^{-1} = \log_y x$, (1a) can be rewritten as

$$\gamma + \gamma^{-1} = \frac{26}{5}\,, \tag{2}$$

where

$$\gamma \equiv \log_x y\,. \tag{3}$$

But (2) can be written in standard form as

$$\gamma^2 - \frac{26}{5}\gamma + 1 = 0\,, \tag{4}$$

with solutions

$$\gamma_> = 5\,, \qquad \gamma_< = \frac{1}{5}\,. \tag{5}$$

Now we need (1b), to which we apply $\log_x$ to both sides, getting

$$\log_x x + \log_x y = \log_x 64\,, \tag{6}$$

or

$$1 + \log_x y = 1 + \gamma_> = 1 + 5 = 6 = \log_x 64\,, \tag{7}$$

On dividing through by 6 and adjusting a bit, we have that

$$1 = \log_x 64^{1/6} = \log_x 2\,, \tag{8}$$

After raising x to this last equation, we get

$$x = 2 \quad \text{and thus} \quad y = 32\,, \tag{9}$$

where we used (1b).

And if we use $\gamma_<$ in (7) instead of $\gamma_>$, we will get

$$x = 32 \quad \text{and thus} \quad y = 2\,, \tag{10}$$

which should come as no surprise, as x and y entered the problem symmetrically.