Structured Differentiation for Advanced Calculus

P. Reany

May 8, 2025

> Matters of notation play a
> considerable role in connection with
> the chain rule. Wide varieties
> of usage exist in mathematical writing
> where the chain rule is concerned.
>
> —Taylor & Mann

Comment: I'm here republishing this article in smaller sections for this web format. Partial differentiation is a highly nontrivial subject. So let's try to make it make sense. Knowledge of how to multiply matrices together and how to take the multiplicative inverse of a 2×2 matrix is necessary.

Introduction

The purpose of this paper is to present a structured, semantically unified formalism for differentiation to meet the needs of the undergraduate and graduate mathematics student. The problems in this paper are taken mostly from texts on advanced calculus.

Assorted problems from various sources

PROBLEM:

Let $w(\mathbf{x}) = w(r(\mathbf{x}), \phi(\mathbf{x}))$ be given by $\mathbf{x} = (x, y)^t$ and

$$\begin{cases} x = r\cosh\phi \\ y = r\sinh\phi \end{cases} . \tag{1}$$

Find the derivatives of w with respect to x, y in terms of derivatives of r, ϕ.

SOLUTION:

There are two simple ways to do this. In both cases we wish to write the equation

$$\frac{\partial w}{\partial \mathbf{x}} = \frac{\partial w}{\partial \mathbf{u}} \frac{\partial \mathbf{u}}{\partial \mathbf{x}}\,, \tag{2}$$

where $\mathbf{u} = (r(\mathbf{x}), \phi(\mathbf{x}))^t$. So what we need to know is the transformation matrix $\frac{\partial \mathbf{u}}{\partial \mathbf{x}}$. The first way to get it to write the equation $\mathbf{x} = \mathbf{x}(\mathbf{u}(\mathbf{x}))$. Then the total derivative of this by $\mathbf{x}$ gives us

$$1 = \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \frac{\partial \mathbf{x}}{\partial \mathbf{u}}\,, \tag{3}$$

from which we get

$$\frac{\partial \mathbf{u}}{\partial \mathbf{x}} = \left(\frac{\partial \mathbf{x}}{\partial \mathbf{u}}\right)^{-1} . \tag{4}$$

The whole idea here is that the derivative we want to calculate are on the LHS of (2), but are expressed in terms of $\partial \mathbf{u}/\partial \mathbf{x}$; but to perform these derivatives directly, we would first need to algebraically solve for r and ϕ in terms of x and y. Yuck! We already have x and y in terms of r and ϕ, so why can't we use that? Well, we can — and that's the whole point of this exercise.

The second way is to virtually emplace $\mathbf{F} = 0$ by the equations

$$\begin{cases} F_1 = x - r\cosh\phi \\ F_2 = y - r\sinh\phi \end{cases} \tag{5}$$

where the subscripts are not indicating partial derivatives. Then we just use the formula

$$\frac{\delta \mathbf{F}}{\delta \mathbf{x}} = \frac{\partial \mathbf{F}}{\partial \mathbf{x}} + \frac{\partial \mathbf{F}}{\partial \mathbf{u}} \frac{\partial \mathbf{u}}{\partial \mathbf{x}} = \mathbf{0}\,, \tag{6}$$

In either case we end up with

$$\left(\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}\right) = \left(\frac{\partial w}{\partial r}, \frac{\partial w}{\partial \phi}\right) \begin{pmatrix} \cosh\phi & -\sinh\phi \\ \frac{-\sinh\phi}{r} & \frac{\cosh\phi}{r} \end{pmatrix}, \tag{7}$$

which, of course, is valid where r is different from zero.

DEFINITION: A *convolution* is said to occur whenever a variable or function is dependent on itself (this is not directly related to convolution in Laplace transform theory).

EXAMPLE:

For an example, consider the equation

$$f(f(x,y),y) = 0\,. \tag{8}$$

Now say that we want to solve for y as a function of x about some point p. We may derive yet another condition on f by differentiating (8) by y.

$$\frac{\delta f}{\delta y} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial x} \frac{\partial x}{\partial y} = 0\,, \tag{9}$$

but since $x = f(x, y)$ then $\partial x/\partial y = \partial f/\partial y$, so

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} = 0\,. \tag{10}$$

Factoring, we get

$$\frac{\partial f}{\partial y}\left(1 + \frac{\partial f}{\partial x}\right) = 0\,. \tag{11}$$

But at p, $\partial f/\partial y \neq 0$ by assumption, therefore

$$\frac{\partial f}{\partial x} = -1\,. \tag{12}$$