Math Diversion Problem 570

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My [algebraic] methods are really methods of working
and thinking; this is why they have crept in
everywhere anonymously.
— Emmy Noether

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=WBpxl44Egzk
Title: Can you solve this Tricky Exponential Algebra Question?
 Presenter: Maths & Olympiad
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1 The Problem

Given the relation

$$x^{5x^{95}} = 1444\,, \tag{1}$$

find all real values of x.

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{2}$$

then

$$z = W(B)\,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

A lemma I'll need from the theory of the Lambert W function is the following: If

$$y \ln y = B\,, \tag{4}$$

then

$$\ln y = W(y \ln y) = W(B)\,. \tag{5}$$

The following is the 'Lambert W function base s'[1], or W_s, where s is a positive real number. Let's begin with the relation

$$xs^x = A\,, \tag{6}$$

which looks very similar to (2). Then

$$x = W_s(xs^x) \equiv \frac{W(A \ln s)}{\ln s}\,. \tag{7}$$

But when $s = e$, we have that

$$x = W_e(xe^x) = \frac{W(A \ln e)}{\ln e} = W(A)\,, \tag{8}$$

which is the usual Lambert W function. (By the way, the proof to this lemma is not hard. It begins with setting $s^x = e^y$ and proceeding from there.)

If s is an integer, I may resort to putting parentheses around it to distinguish it from the n-series, as such $W_{(s)}$.

3 The Solution

(Note: $1444 = 38^2$.)

Let's begin with an α substitution:

$$x = 38^{\alpha}\,. \tag{9}$$

Then the Given becomes

$$(38^{5\alpha})^{38^{95\alpha}} = 38^2\,, \tag{10}$$

On equating exponents, we get

$$5\alpha \cdot 38^{95\alpha} = 2\,. \tag{11}$$

Next, multiply through by 19:

$$95\alpha \cdot 38^{95\alpha} = 38\,. \tag{12}$$

Now we can use one of the Lambert lemmas above, to get

$$95\alpha = W_{(38)}(38) = \frac{W_n(38 \cdot \ln 38)}{\ln 38}\,. \tag{13}$$

On setting $n = 0$, we have that

$$95\alpha_0 = \frac{W_0(38 \cdot \ln 38)}{\ln 38} = \frac{\ln 38}{\ln 38} = 1\,. \tag{14}$$

[1] This notation I invented myself.

Therefore, for the principal value of alpha, we have

$$\alpha_0 = \frac{1}{95}\,. \tag{15}$$

On substituting this into (9), we have, finally,

$$x = 38^{1/19}\,. \tag{16}$$