Math Diversion 572

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The Axiom of Choice is obviously true; the Well Ordering
Principle is obviously false; and who can tell
about Zorn's Lemma?
— Jerry Bona

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Source: https://www.youtube.com/watch?v=1Sx3Fn-MRQw
Title: THE DERIVATIVE OF THE ARCOTANGENT.
 Presenter: Matematicas con Juan
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1 The Problem

Given the relation

$$\tan\left(\tan^{-1} x\right) = x\,, \tag{1}$$

find

$$\frac{d}{dx}\tan^{-1} x\,. \tag{2}$$

2 The Preparation

Some useful identities:

$$\frac{d}{dx}\tan x = \sec^2 x\,. \tag{3a}$$

$$\sec x = \frac{1}{\cos x}\,. \tag{3b}$$

3 The Solution

If there's an inverse tangent, there's got to be an angle somewhere inside a triangle, which we'll call θ, that corresponds to it. Let's put it inside a right triangle and see what goes where.

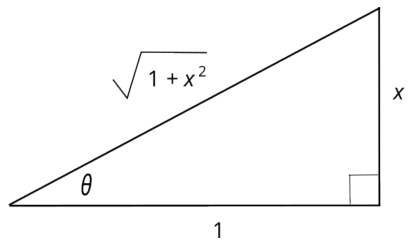

Figure 1. We get to create this right triangle to our liking. We are free to setup the triangle such that $\tan\theta = x/1 = x$, so that $\theta = \tan^{-1} x$.

Next, we differentiate (1), applying the chain rule as we go:

$$\frac{d}{dx}\tan\left(\tan^{-1} x\right) = \frac{d}{d\theta}\tan\theta\Big|_{\theta=\tan^{-1}x}\frac{d}{dx}\left(\tan^{-1} x\right) = 1\,, \tag{4}$$

or

$$\sec^2\theta\Big|_{\theta=\tan^{-1}x}\frac{d}{dx}\left(\tan^{-1} x\right) = 1\,, \tag{5}$$

or

$$\sec^2(\tan^{-1} x)\,\frac{d}{dx}\left(\tan^{-1} x\right) = 1\,. \tag{6}$$

Solving for $\frac{d}{dx}(\tan^{-1} x)$, we have that

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \cos^2\left(\tan^{-1} x\right). \tag{7}$$

Now,

$$\cos\left(\tan^{-1} x\right) = \cos\,\theta = \frac{1}{\sqrt{1+x^2}}\,. \tag{8}$$

And finally,

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}\,. \tag{9}$$