Structured Differentiation for Advanced Calculus

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> Matters of notation play a considerable role in connection with the chain rule. Wide varieties of usage exist in mathematical writing where the chain rule is concerned.
>
> —Taylor & Mann

Comment: I'm here republishing this article in smaller sections for this web format. Partial differentiation is a highly nontrivial subject. So let's try to make it make sense. Knowledge of how to multiply matrices together and how to take the multiplicative inverse of a 2×2 matrix is necessary.

Introduction

The purpose of this paper is to present a structured, semantically unified formalism for differentiation to meet the needs of the undergraduate and graduate mathematics student. The problems in this paper are taken mostly from texts on advanced calculus.

Assorted problems from various sources

PROBLEM:

Let $\mathbf{x} = \mathbf{x}(\mathbf{u})$ where $\mathbf{x}$ is the new fundamental, $\mathbf{u}$ is the old fundamental, $\mathbf{x} = (x, y, z)^t$, $\mathbf{u} = (u, v, w)^t$, show that

$$\frac{\partial x}{\partial u} = \left| \frac{\partial(v, w)}{\partial(y, z)} \right| \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| . \tag{1}$$

SOLUTION:

There are likely many good ways to solve this problem. I'll present just one. Let us form the state vector

$$\Psi = \begin{pmatrix} x \\ v \\ w \end{pmatrix} . \tag{2}$$

Now let's introduce some notation. Let $A((\mathbf{x}))$ mean that $\mathbf{x}$ is the fundamental of A, though it may not be the variant of A. Thus, we can write the two parametrizations of Ψ as

$$\Psi((\mathbf{u})) = \begin{pmatrix} x((\mathbf{u})) \\ v \\ w \end{pmatrix}, \qquad \Psi((\mathbf{x})) = \begin{pmatrix} x \\ v((\mathbf{x})) \\ w((\mathbf{x})) \end{pmatrix}. \tag{3}$$

Now we take the total derivative of $\Psi((\mathbf{u})) = \Psi(\mathbf{x}(\mathbf{u}))$ by $\mathbf{u}$ and simplify (assuming that $x = x(\mathbf{u})$, $v = v(\mathbf{x})$, and $w = w(\mathbf{x})$), using the chain rule on the RHS, to get

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{pmatrix} \left(\frac{\partial \mathbf{x}}{\partial \mathbf{u}} \right). \tag{4}$$

And on taking the determinant of both sides of this we get (1).

PROBLEM:

Let $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. Find $\partial z/\partial x$ and $\partial z/\partial y$.

SOLUTION:

One way to solve this is to solve for z directly, but his would introduce messy square roots. So we'll use implicit differentiation as we have been doing. Let

$$F = x^2/a^2 + y^2/b^2 + z^2/c^2 - 1 = 0\,. \tag{5}$$

Now let $\boldsymbol{\eta} = (x, y)^t$, then

$$F(\boldsymbol{\eta}, (z(\boldsymbol{\eta})) = 0\,. \tag{6}$$

On differentiating this we get

$$\frac{\delta F}{\delta \boldsymbol{\eta}} = \frac{\partial F}{\partial \boldsymbol{\eta}} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial \boldsymbol{\eta}} = 0\,. \tag{7}$$

Solving this for $\partial z/\partial \boldsymbol{\eta}$, we get

$$\frac{\partial z}{\partial \boldsymbol{\eta}} = -\left(\frac{\partial F}{\partial z} \right)^{-1} \frac{\partial F}{\partial \boldsymbol{\eta}}\,. \tag{8}$$

From which we get

$$\left(\frac{\partial z}{\partial x}, \quad \frac{\partial z}{\partial y} \right) = -\left(\frac{2z}{c^2} \right)^{-1} \left(\frac{2x}{a^2}, \quad \frac{2y}{b^2} \right) = \left(-\frac{xc^2}{za^2}, \quad -\frac{yc^2}{zb^2} \right). \tag{9}$$