

Math Diversion Problem 574

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Your work has changed the way we look at algebra...No-one,
perhaps, contributed as much as you towards remoulding
the axiomatic approach into a powerful research
instrument... Amongst your predecessors in
algebra and number theory it was probably
Dedekind who came closest.
— Hermann Weyl's speech commemorating
the life work of Emmy Noether

The YouTube video is found at:

Source: <https://www.youtube.com/watch?v=skK2GefAgg8>

Title: Logarithm & Algebra

Presenter: Maths & Olympiad

1 The Problem

Given the relation

$$x = 4^{\log_8 \sqrt{18\sqrt{5}-17}}, \quad (1)$$

simplify x and then find

$$\phi = \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \dots. \quad (2)$$

2 The Solution

First,

$$x = 4^{\log_8 \sqrt{18\sqrt{5}-17}} = 2^{2 \log_8 \sqrt{18\sqrt{5}-17}} = 2^{\log_8 (18\sqrt{5}-17)}. \quad (3)$$

Now, take the logarithm base 2:

$$\log_2 x = \log_8 (18\sqrt{5} - 17) \quad (4)$$

$$\frac{\log x}{\log 2} = \frac{\log (18\sqrt{5} - 17)}{\log 8} = \frac{\log (18\sqrt{5} - 17)}{\log 2^3} = \frac{\log (18\sqrt{5} - 17)}{3 \log 2}, \quad (5)$$

therefore,

$$\log x = \frac{1}{3} \log (18\sqrt{5} - 17) = \log (18\sqrt{5} - 17)^{1/3}, \quad (6)$$

And finally,

$$x = (18\sqrt{5} - 17)^{1/3}. \quad (7)$$

Now for ϕ . First, let

$$\frac{1}{x} = y. \quad (8)$$

Note that $|\frac{1}{x}| < 1$ therefore $|y| < 1$. Then,

$$\phi + 1 = 1 + y + y^2 + y^3 + \dots \quad (9)$$

$$= \sum_{k=0}^{\infty} y^k = \frac{1}{1-y} = \frac{1}{1-x^{-1}} = \frac{x}{x-1}. \quad (10)$$

So,

$$\phi = \frac{x}{x-1} - 1 = \frac{1}{x-1} = \frac{1}{(18\sqrt{5} - 17)^{1/3} - 1}. \quad (11)$$