Math Diversion Problem 575

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> Abstract algebra, as a conscious discipline, starts with Noether's 1921 paper "Ideal Theory in Rings."
> — Saunders MacLane

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=I8X2pMJlD9Q
Title: Unleashing Your Math Skills:
 Presenter: Numbers & Numbers
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1 The Problem

Given the relation

$$x^x = 7^{x+49}\,, \tag{1}$$

find all real values of x.

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{2}$$

then

$$z = W(B)\,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

A lemma I'll need from the theory of the Lambert W function is the following: If

$$y \ln y = B\,, \tag{4}$$

then

$$\ln y = W(y \ln y) = W(B)\,. \tag{5}$$

The following is the 'Lambert W function base s'[1], or W_s, where s is a positive real number. Let's begin with the relation

$$xs^x = A\,, \tag{6}$$

which looks very similar to (2). Then

$$x = W_s(xs^x) \equiv \frac{W(A\ln s)}{\ln s}\,. \tag{7}$$

But when $s = e$, we have that

$$x = W_e(xe^x) = \frac{W(A\ln e)}{\ln e} = W(A)\,, \tag{8}$$

which is the usual Lambert W function. (By the way, the proof to this lemma is not hard. It begins with setting $s^x = e^y$ and proceeding from there.)

If s is an integer, I may resort to putting parentheses around it to distinguish it from the n-series, as such $W_{(s)}$.

3 The Solution

Let's begin with an α substitution:

$$x = 7^\alpha\,. \tag{9}$$

Then the Given becomes

$$7^{\alpha 7^\alpha} = 7^{7^\alpha + 49}\,. \tag{10}$$

On equating exponents, we get

$$\alpha 7^\alpha = 7^\alpha + 7^2\,. \tag{11}$$

After a little algebra, we get

$$(\alpha - 1)7^{(\alpha-1)} = 7\,. \tag{12}$$

Now we can use one of the Lambert lemmas above, to get

$$\alpha - 1 = W_{(7)}(7) = \frac{W(7\cdot\ln 7)}{\ln 7} = \frac{\ln 7}{\ln 7} = 1\,, \tag{13}$$

where we use the other Lambert lemma. Hence,

$$x = 7^2 = 49\,. \tag{14}$$

[1] This notation I invented myself.