Math Diversion Problem 577

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> Young men should prove theorems, old
> men should write books.
> — G.H. Hardy

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=xuugqqgG6d4
Title: Oxford entrance exam question
 Presenter: Math Beast
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1 The Problem

Given the relation

$$x^{1/x} = e^{\pi/2}\,, \tag{1}$$

find all values of x.

Note: Since I've seen the answer given by WolframAlpha, I must conclude that this problem is beyond my ability in numerical analysis of the details. However, I think that its solution to the principal value is very close to what I get. WolframAlpha presents both i and $-i$ as approximations, their real parts being exceedingly small.

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{2}$$

then

$$z = W(B)\,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

Note:

$W_0(-\pi/2) = i\pi/2$.

A lemma I'll need from the theory of the Lambert W function is the following: If

$$y \ln y = B\,, \tag{4}$$

then

$$\ln y = W(y \ln y) = W(B)\,. \tag{5}$$

The following is the 'Lambert W function base s'[1], or W_s, where s is a positive real number. Let's begin with the relation

$$xs^x = A\,, \tag{6}$$

which looks very similar to (2). Then

$$x = W_s(xs^x) \equiv \frac{W(A \ln s)}{\ln s}\,. \tag{7}$$

But when $s = e$, we have that

$$x = W_e(xe^x) = \frac{W(A \ln e)}{\ln e} = W(A)\,, \tag{8}$$

which is the usual Lambert W function. (By the way, the proof to this lemma is not hard. It begins with setting $s^x = e^y$ and proceeding from there.)

If s is an integer, I may resort to putting parentheses around it to distinguish it from the n-series, as such $W_{(s)}$.

3 The Solution

Let's begin by defining

$$\beta = e^{\pi/2}\,. \tag{9}$$

Then on raising (1) to the x'th power, we get

$$x = \beta^x\,. \tag{10}$$

With a little algebra, this becomes

$$-x\beta^{-x} = -1\,. \tag{11}$$

Now we can use one of the Lambert lemmas above, to get

$$-x = W_\beta(-1) = \frac{W_n(-1 \cdot \ln \beta)}{\ln \beta} = \frac{W_n(-\pi/2)}{\pi/2}\,. \tag{12}$$

On setting $n = 0$, we have that

$$-x_0 = \frac{W_0(-\pi/2)}{\pi/2} = \frac{i\pi/2}{\pi/2} = i\,. \tag{13}$$

Therefore, for the principal value of x, we have

$$x_0 = -i\,. \tag{14}$$

[1]This notation I invented myself.