Math Diversion Problem 583

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> Beauty is the first test: there is no permanent place
> in the world for ugly mathematics.
> — G.H. Hardy

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=EZO2Eqoew2A
Title: Can You Solve This?
 Presenter:  Brain Station
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1 The Problem

Given the relations

$$x^2 + y^2 = 7\,, \tag{1a}$$
$$x^3 + y^3 = 10\,, \tag{1b}$$

find the values

$$\phi = x + y\,, \tag{2}$$

over the reals.

2 The Solution

$$\phi^2 = x^2 + 2xy + y^2 = 7 + 2xy \quad (\text{from (1a)})\,, \tag{3a}$$
$$\phi^3 = x^3 + 3x^2y + 3xy^2y^3 = 10 + 3xy(x + y) \quad (\text{from (1b)})\,. \tag{3b}$$

Solving (3a) for xy and then substituting that into (3b) and applying a bit of algebra, we have that

$$\phi^3 - 21\phi + 20 = 0\,. \tag{4}$$

Hopefully, we can one root by use of a table.

ϕ	$\phi^3 - 21\phi + 20$
3	$27 - 63 + 20 = 16$
4	$64 - 84 + 20 = 0$ ✓

Table 1: Heuristic: One root solved by Table.

Now that we know that one root is $\phi = 4$, we can employ long division to get

$$(\phi - 4)(\phi^2 + 4\phi - 5) = 0\,. \tag{5}$$

Using the quadratic formula to find the other two roots, we get

$$\phi = -5,\ 1\,. \tag{6}$$