Math Diversion Problem 584

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> All my life I kept running into smart people.... In school there were lots of smarter kids. And when I first joined the force, sir, they had some very clever people there. And I could tell right away that it wasn't going to be easy making detective as long as they were around. What I figured that... if I worked harder than they did. Put in more time. Read the books. Kept my eyes open. Maybe I could make it happen. And I did!
> — Lt. Columbo to his prisoner
> (from the TV show *Columbo*)
> ("The Bye Bye Sky High)
> IQ Murder Case")

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=2JJOd4ZgeEU
Title: Can You Dare To Touch This?
 Presenter: Brain Station Advanced
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1 The Problem

Given the relation

$$F(x, y) = x^y \,, \tag{1}$$

which satisfies constraint

$$x + y = 6 \,, \tag{2}$$

find all values of x which maximize F.

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B \,, \tag{3}$$

then

$$z = W(B)\,, \tag{4}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

Note:

$W_0(-\pi/2) = i\pi/2$.

A lemma I'll need from the theory of the Lambert W function is the following: If

$$y \ln y = B\,, \tag{5}$$

then

$$\ln y = W(y \ln y) = W(B)\,. \tag{6}$$

The following is the 'Lambert W function base s'[1], or W_s, where s is a positive real number. Let's begin with the relation

$$xs^x = A\,, \tag{7}$$

which looks very similar to (3). Then

$$x = W_s(xs^x) \equiv \frac{W(A \ln s)}{\ln s}\,. \tag{8}$$

But when $s = e$, we have that

$$x = W_e(xe^x) = \frac{W(A \ln e)}{\ln e} = W(A)\,, \tag{9}$$

which is the usual Lambert W function. (By the way, the proof to this lemma is not hard. It begins with setting $s^x = e^y$ and proceeding from there.)

If s is an integer, I may resort to putting parentheses around it to distinguish it from the n-series, as such $W_{(s)}$.

One last result we'll need is

$$\beta = W_n(\beta) e^{W_n(\beta)}\,. \tag{10}$$

3 The Solution

Let's begin by defining a new function $f(x)$ by

$$f(x) = x^{6-x}\,, \tag{11}$$

[1] This notation I invented myself.

where we replaced y by $6-x$. So, what we need to maximize is $f(x)$, which we can by setting its derivative by x to zero, i.e., $f'(x)=0$.

However, the standard way to proceed when we have functional exponents is to use what's referred to as 'logarithmic differentiation'. So, first, we take the logarithm across (11) and then differentiate that.

$$\ln f(x) = (6-x)\ln x\,, \tag{12}$$

then, on differentiating, we get,

$$\frac{f'(x)}{f(x)} = -\ln x + \frac{6-x}{x}\,, \tag{13}$$

On setting $f'(x)=0$, (13) becomes

$$\ln x = \frac{6-x}{x}\,. \tag{14}$$

Now, let

$$x = y^{-1}\,, \tag{15}$$

so that (14) becomes

$$\ln y^{-1} = 6y - 1\,. \tag{16}$$

On raising e to this last equation, ewe get

$$y^{-1} = e^{6y}/e\,, \tag{17}$$

which can be rewritten as

$$6ye^{6y} = 6e\,. \tag{18}$$

On taking the Lambert W function across this, we have that

$$6y = W_0(6e)\,, \tag{19}$$

where we have indicated that our interest is in real values of the W function. Now, returning to variable x:

$$\frac{6}{x} = W_0(6e) = W(6e)\,, \tag{20}$$

Now, from (10), with $n=0$ and $\beta = 6e$:

$$6e = W(6e)e^{W(6e)}\,. \tag{21}$$

where we dropped the subscript on W. On using

$$W(6e) = \frac{6e}{e^{W(6e)}} \tag{22}$$

in (20), we get

$$\frac{1}{x} = \frac{e}{e^{W(6e)}}\,, \tag{23}$$

Finally, on inverting this, we have that

$$x = e^{W(6e)-1}\,. \tag{24}$$