Math Diversion Problem 590

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> Easy to criticize, more difficult to be correct.
> — Charlie Chan

1 Introduction

This is the first of a large collection of word problems that I published some years ago. My sources were from textbooks and Internet posts. I came to describe the method of organization and solution that I applied to these algebra word problems as **Scheme**.

2 The Problem

In one of my chemistry textbooks [1], I find this problem (presented here somewhat paraphrased):

> What is the $[\mathrm{H}^+]$ in 0.50 M $\mathrm{HC_2H_3O_2}$ solution? The ionization constant K_a for $\mathrm{HC_2H_3O_2}$ is 1.8×10^{-5}.
>
> Given the equilibrium state of the ionization reaction (for acetic acid)
>
> $$\mathrm{HC_2H_3O_2} \rightleftharpoons \mathrm{H}^+ + \mathrm{C_2H_3O_2}^- \,, \tag{1}$$
>
> the ionization constant given to us as
>
> $$K_a = \frac{[\mathrm{H}^+]\,[\mathrm{C_2H_3O_2}^-]}{[\mathrm{HC_2H_3O_2}]} = 1.8 \times 10^{-5} \,. \tag{2}$$

What Eq. (1) is telling us is that the acetate molecule in water only weakly ionizes, and so it will exist in equilibrium with its ionized cousin. Eq. (2) provides for us a mathematical relationship by which we can solve for the amount of ionization.

Just as an observation, we can use this last equation to solve for $[\mathrm{H}^+]$ if we can express $[\mathrm{C_2H_3O_2}^-]$ and $[\mathrm{HC_2H_3O_2}]$ in terms of a single variable (this is standard algebra: one equation needs one variable in it to solve for). The means to accomplish this goal is the essence of how to complete the problem. However, it's ironic that the worst place to begin to solve this problem may be to actually ask oneself "Is there a way to write $[C_2H_3O_2{}^-]$ and $[HC_2H_3O_2]$ in terms of a single variable?" Instead, we begin with the following question: "Are there any totals or parts suggested in this problem?" This one question will be central to how we will begin to solve most algebra problems in this series of papers.

3 Solution Part 1: Conceptualizing the Problem

Before I can write down an algebraic equation (extracted from a word problem) I can solve (unless it's just plain obvious), I must first understand the problem in terms of my heuristics – my *Scheme* of solving algebra word problems I cobbled together decades ago. For instance, are there any totals about, or parts lying around? Why is that a useful question? Because every total is equal to the sum of its parts. What good is that? Well, it's an equation, and algebra is the art and technique of solving one or more nondifferential equations simultaneously:

$$\text{Every Total} = \sum_i \text{Parts}_i \, . \tag{3}$$

Another important equation type comes from identifying some invariant in a before-and-after process. I propose to create this (weak) acetic acid solution by employing an idealized before-and-after process. Suppose we begin with 0.50 moles of pure (anhydrous) $HC_2H_3O_2$. Then we add just enough water to the anhydrous acetic acid to create exactly 1 liter of solution and give time for the system to come to equilibrium. Why? Because that's when we are allowed to invoke Eq. (2).[1]

We started off with 0.50 moles of acetic acid molecules ($HC_2H_3O_2$), but we didn't end up with 0.50 moles of it. So what, if anything, was the invariant of the process? To answer this, I need to stipulate a new term. An *acetate body* exists in one of two forms: either the acetate ion $C_2H_3O_2{}^-$ or the acetic acid molecule $HC_2H_3O_2$. Our symbol for the generic acetate body will be $<C_2H_3O_2>$. Every acetate body will either be bound or unbound to a hydrogen ion H^+ (or proton). So, the correspondences are:

$$<C_2H_3O_2>_b \sim HC_2H_3O_2 \quad \text{and} \quad <C_2H_3O_2>_u \sim C_2H_3O_2{}^- \, . \tag{4}$$

In other words, the basic assumption of this problem is that the number of acetate bodies is the main invariant of this before-and-after process of adding the pure water to the anhydrous acetic acid.

$$\text{total \# acetate bodies before} = \text{total \# acetate bodies after} \, . \tag{5}$$

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[1] I'm not saying that this is the process you can or will use in the lab to obtain such a substance. The purpose of conceptualizing the problem this way is to setup a before-and-after situation because there will always be something that is conserved from the Before-State to the After-State, and that implies the existence of an equation to solve for something in some variable, say Y, hence: $Y_{\text{Before}} = Y_{\text{After}}$.

In the before state, the acetate bodies were all in the bound state, but all we care about is that their total was 0.50 moles. In the after state, the total acetate bodies were the sum of the bound acetate bodies (that is, the unionized acetate) and the unbound acetate bodies (that is, the ionized acetate). Thus (5) becomes (in moles)

$$\begin{aligned} 0.50 &= \#\!< C_2H_3O_2 >_b + \#\!< C_2H_3O_2 >_u \\ &= \#(HC_2H_3O_2) + \#(C_2H_3O_2{}^-)\,, \end{aligned} \tag{6}$$

where all terms are in moles. Now we divide through by 1 liter, and then the units go from moles to moles/liter, or molarity M:

$$\begin{array}{lccccc} & [HC_2H_3O_2]_{\text{Before}} & = & [HC_2H_3O_2]_{\text{After}} & + & [\ C_2H_3O_2{}^-]_{\text{After}} \\ \text{Molarity:} & 0.50 & = & (0.50-x) & + & x\,, \end{array} \tag{7}$$

where $[HC_2H_3O_2]_{\text{After}}$ is the concentration of undissociated acetic acid in the 'after state', or, in more familiar chemistry terms, it is the concentration of undissociated acetic acid in the solution at equilibrium.

The simplest way to proceed is to set $[\ C_2H_3O_2{}^-]$ to x to substitute into (2). Now, because there is a 1–1 correspondence between the moles of H^+ and the moles of $C_2H_3O_2{}^-$, there is also a 1–1 correspondence between the molarities of H^+ and the moles of $C_2H_3O_2{}^-$; therefore $[H^+] = [\ C_2H_3O_2{}^-] = x$.

Given that I know where this problem is leading, I must interpret Eq. (2) as the following:

$$K_a = \frac{[H^+]_{\text{After}}\,[C_2H_3O_2{}^-]_{\text{After}}}{[HC_2H_3O_2]_{\text{After}}} = 1.8 \times 10^{-5}\,, \tag{8}$$

where $[H^+]_{\text{After}}$ ignores the initial spontaneous ionization of pure water prior to mixing it with the anhydrous acetate.

I remind the reader that the subscripts of "After" used in (8) are not standard. They merely reflect how we modeled the formation of this fluid as a 'before-and-after' process. However, it does remind us that we should be using $0.50 - x$ for $[HC_2H_3O_2]$ in (8) and not using 0.50.

4 Solution Part 2: Solving the quadratic

We've now completed the conceptual part of the solution and we're ready to write down an algebraic equation and solve it analytically.

From (7), we get $[HC_2H_3O_2]_{\text{After}} = 0.50 - x$. We can now rewrite (8) as

$$\frac{x \cdot x}{0.50 - x} = 1.8 \times 10^{-5}\,. \tag{9}$$

Assuming x to be much less than 0.50, we can safely ignore it in the denominator, to get

$$x^2 = (0.50)1.8 \times 10^{-5}\,. \tag{10}$$

or

$$x = 3.0 \times 10^{-3} \text{ mol/liter H}^+ \,. \tag{11}$$

If we opt to solve the more labor intensive quadratic Eq. (9) instead, we get for the extra effort the value

$$x = 2.99 \times 10^{-3} \text{ mol/liter H}^+ \,, \tag{12}$$

which, after rounding, is really the same answer as before.

References

[1] M. Hein and S. Arena *Foundations of College Chemistry*, alternate 12th ed, John Wiley & Sons (2007), 421–422.