Math Diversion Problem 591

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In mathematics, you don't understand things.
You just get used to them.
— John von Neumann

1 Introduction

The purpose of this paper is to present a structured, semantically unified formalism for differentiation to meet the needs of the undergraduate and graduate mathematics student.

2 PROBLEM:

This Internet problem is found at

`https://www.physicsforums.com/threads/partial-differentiation-problem-multiple-variables-chain-rule.753651/`

If

$$z = x^2 + 2y^2\,, \tag{1a}$$
$$x = r\cos\theta\,, \tag{1b}$$
$$y = r\sin\theta\,, \tag{1c}$$

find the partial derivative $\left(\dfrac{\partial z}{\partial \theta}\right)_x$.

Suggested answer:

$$\left(\frac{\partial z}{\partial \theta}\right)_x = 4r^2\tan\theta\,, \tag{2}$$

where convention expects us to consider $\dfrac{\partial z}{\partial \theta}$ as a total derivative[1] and $\left(\dfrac{\partial z}{\partial \theta}\right)_x$ is the total derivative of z by θ, holding x constant in the process.

[1]It's clearly a total derivative by SD nomenclature, anyway.

Now, I must make a minor point of contention with the answer given in (2). My preference is that the only variables that should appear on the RHS should be x and θ.

3 Solution 1:

Before we rush into an attempted solution, it's good to take stock of what we have been given to work with. We are asked to find $\left(\frac{\partial z}{\partial \theta}\right)_x$ as though we know what $z = z(x, \theta)$ is, but we don't. What we actually know is $z = z(x, y)$. There are three ways to proceed at this point.[2] One way to proceed is simply to algebraically recast z accordingly

$$z = z(x, y) \longrightarrow z = z(x, \theta)\,. \tag{3}$$

This part is easy:

$$z(x, \theta) = x^2(1 + 2\tan^2\theta)\,, \tag{4}$$

In SD, we treat $z(x, \theta)$ as primitive in its arguments (i.e., x, θ are the new independent variables of the problem), so that the total derivative by θ taken across this last equation reduces on both sides to the partial derivative by θ, hence:

$$\frac{\partial z}{\partial \theta} = 4x^2 \tan\theta \sec^2\theta = 4r^2 \tan\theta\,, \tag{5}$$

where $\frac{\partial z}{\partial \theta}$ is, of course, an explicit derivative in SD, or rather $\left(\frac{\partial z}{\partial \theta}\right)_x$.

Definition: I have long had trouble thinking that I can change the set of independent variables on which some other variable is dependent. It's probably just a psychological thing. In any case, I prefer to refer to the ordered set of independent variables as the *fundamental.* I usually denote the old fundamental by the vector $\boldsymbol{\eta}$ and the new fundamental by $\boldsymbol{\eta}'$.

Rule 1) When one fundamental variable η_i is totally differentiated by a cofundamental variable η_j, the result is

$$\frac{\delta \eta_i}{\delta \eta_j} = \delta_{ij}\,, \tag{6}$$

where δ_{ij} is the Kronecker delta, meaning that when a fundamental variable is totally differentiated by itself, the result is unity, but when it is totally differentiated by a different cofundamental variable, the result is zero; hence, the notion of *variable independence* between different elements in a given fundamental is captured by this rule. Note: To remove ambiguity, any variable in a fundamental set is cofundamental to itself.[3]

[2] I found three ways, but there could be more.

[3] We will consider it axiomatic in SD that the total derivative of any variable with respect to itself is unity.

Rule 2) In all other cases, i.e., when a new fundamental variable differentiates an old fundamental variable, or vice versa, the deltal derivative usually reduces to an explicit derivative. This is the standard thing to do in thermodynamics because, by default, we regard state variables as having no implicit dependence on other variables (at least this is my knowledge regarding them).[4]

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4 Second Solution:

What we did in the last solution was to algebraically recast $z = z(x, y) \longrightarrow z = z(x, \theta)$ because it was algebraically easy to do. But what if such a transformation is algebraically clumsy or even impossible? Then what do we do? Ans: We proceed as if such a change of fundamental is possible, but use the chain rule instead. Let's see how this works.

The de facto fundamental of z in (1a) is

$$\boldsymbol{\eta} \equiv \begin{bmatrix} x \\ y \end{bmatrix} , \tag{7}$$

which we'll think of as the 'old fundamental'. Just as before, what we'd like is the parameterization of z as $z(x, \theta)$, but this time we're not going to make the algebraic substitution. This time, we're going to set the 'new fundamental' as

$$\boldsymbol{\eta}' \equiv \begin{bmatrix} x \\ \theta \end{bmatrix} . \tag{8}$$

From here, we can think of the problem as a 'change in fundamental' problem, given by

$$z(\boldsymbol{\eta}') = z(\boldsymbol{\eta}(\boldsymbol{\eta}')) . \tag{9}$$

We're going to assume that this change of variables is legitimate. Anyway, differentiating this by $\boldsymbol{\eta}'$, we get by the chain rule

$$\frac{\delta z}{\delta \boldsymbol{\eta}'} = \frac{\partial z}{\partial \boldsymbol{\eta}} \frac{\delta \boldsymbol{\eta}}{\delta \boldsymbol{\eta}'} . \tag{10}$$

Now, remembering that the z on the LHS is primitive in $\boldsymbol{\eta}'$, the total derivative can be replaced by an explicit derivative. Therefore, we have that

$$\frac{\partial z}{\partial \boldsymbol{\eta}'} = \frac{\partial z}{\partial \boldsymbol{\eta}} \frac{\delta \boldsymbol{\eta}}{\delta \boldsymbol{\eta}'} . \tag{11}$$

Expanding this, we get

$$\left[\frac{\partial z}{\partial x}, \frac{\partial z}{\partial \theta}\right] = \left[\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right] \begin{bmatrix} \frac{\delta x}{\delta x} & \frac{\delta x}{\delta \theta} \\ \frac{\delta y}{\delta x} & \frac{\delta y}{\delta \theta} \end{bmatrix} . \tag{12}$$

[4] When a state variable **does** have an implicit dependence on some variable, then we cannot just drop the implicit derivative as we have done in this case.

But, in the 2×2 matrix above, $\dfrac{\delta x}{\delta x} = 1$ and $\dfrac{\delta x}{\delta \theta} = 0$, where this assignment is demanded because of (8), which treats x and θ as mutually independent of each other. So (12) becomes

$$\left[\frac{\partial z}{\partial x}, \frac{\partial z}{\partial \theta}\right] = \left[\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right] \begin{bmatrix} 1 & 0 \\ \dfrac{\delta y}{\delta x} & \dfrac{\delta y}{\delta \theta} \end{bmatrix}$$
$$= [2x\,, 4y] \begin{bmatrix} 1 & 0 \\ \tan\theta & x \sec^2\theta \end{bmatrix}. \tag{13}$$

From this we get to

$$\frac{\partial z}{\partial x} = 2x + 4y \tan\theta\,, \tag{14a}$$
$$\frac{\partial z}{\partial \theta} = 4yx \sec^2\theta\,. \tag{14b}$$

On eliminating y, we get

$$\frac{\partial z}{\partial x} = 2x + 4x \tan^2\theta\,, \tag{15a}$$
$$\frac{\partial z}{\partial \theta} = 4x^2 \tan\theta \sec^2\theta\,. \tag{15b}$$

And for those who prefer the subscripts,

$$\left(\frac{\partial z}{\partial x}\right)_\theta = 2x + 4x \tan^2\theta\,, \tag{16a}$$
$$\left(\frac{\partial z}{\partial \theta}\right)_x = 4x^2 \tan\theta \sec^2\theta\,. \tag{16b}$$

And to make connection to the original form in (5), then

$$\left(\frac{\partial z}{\partial \theta}\right)_x = 4r^2 \tan\theta\,. \tag{17}$$

5 Afterwords:

So, what the point of the second solution? The point is that we have once again verified the methods of SD to provide a systematic approach to these problems, while maintaining the distinction between the new and old fundamentals. This is particularly useful to building an algorithm for an automated partial differentiation system in software.