Math Diversion 592

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With me, everything turns into mathematics.
— Rene Descartes
(P.S. I calculate; therefore I am.)

1 The Problem

On page 93 of NFCM [1], we find problem (6.6): Find the point of intersection of the line defined by the set of all $\mathbf{x}$ satisfying the equation

$$(\mathbf{x}-\mathbf{a})\wedge\mathbf{u}=0\,, \tag{1}$$

and the plane defined by the set of all points $\mathbf{y}$ defined by

$$(\mathbf{y}-\mathbf{b})\wedge\mathbf{B}=0\,, \tag{2}$$

where $\mathbf{u}\wedge\mathbf{B}\neq 0$.

2 Solution

We begin by assuming that there is a unique intersection point $\mathbf{p}$.[1] That is $\mathbf{x}=\mathbf{y}=\mathbf{p}$. Then, from (1)

$$(\mathbf{p}-\mathbf{a})\wedge\mathbf{u}=0 \quad\Longrightarrow\quad (\mathbf{p}-\mathbf{a})\cdot\mathbf{u}=(\mathbf{p}-\mathbf{a})\mathbf{u}=\mathbf{u}(\mathbf{p}-\mathbf{a})\,. \tag{3}$$

And from (2)

$$(\mathbf{p}-\mathbf{b})\wedge\mathbf{B}=0 \quad\Longrightarrow\quad \mathbf{p}\wedge\mathbf{B}=\mathbf{b}\wedge\mathbf{B}\,. \tag{4}$$

Virtually emplacing the vector $\mathbf{a}$ into the LHS of this last equation, gives[2]

$$\mathbf{p}\wedge\mathbf{B}=[(\mathbf{p}-\mathbf{a})+\mathbf{a}]\wedge\mathbf{B}=(\mathbf{p}-\mathbf{a})\wedge\mathbf{B}+\mathbf{a}\wedge\mathbf{B}=\mathbf{b}\wedge\mathbf{B}\,. \tag{5}$$

After applying a bit of algebra to this last equation, we get

$$(\mathbf{p}-\mathbf{a})\wedge\mathbf{B}=(\mathbf{b}-\mathbf{a})\wedge\mathbf{B}\,. \tag{6}$$

[1] Actually, we can infer this from the assumption that $\mathbf{u}\wedge\mathbf{B}\neq 0$.
[2] The point of this inclusion is to construct some kind of relationship between (1) and (2).

From this point on, the goal is to isolate $(\mathbf{p}-\mathbf{a})$, which will make it easy to solve for $\mathbf{p}$.

Now, dotting through on the left by $\mathbf{u}$, gives us

$$\mathbf{u}\cdot[\,(\mathbf{p}-\mathbf{a})\wedge\mathbf{B}\,]=\mathbf{u}(\mathbf{b}-\mathbf{a})\wedge\mathbf{B}=(\mathbf{b}-\mathbf{a})\wedge\mathbf{B}\,\mathbf{u}\,, \tag{7}$$

where we are able to drop the center dot because

$$\mathbf{u}\wedge[\,(\mathbf{p}-\mathbf{a})\wedge\mathbf{B}\,]\equiv\mathbf{0}\,, \tag{8}$$

and where $\mathbf{u}$ commutes with the pseudoscalar $(\mathbf{b}-\mathbf{a})\wedge\mathbf{B}$. However, expanding on the left of (7) gives us

$$\mathbf{u}\cdot(\mathbf{p}-\mathbf{a})\mathbf{B}-(\mathbf{p}-\mathbf{a})\mathbf{u}\cdot\mathbf{B}=(\mathbf{b}-\mathbf{a})\wedge\mathbf{B}\,\mathbf{u}\,, \tag{9a}$$

or rather,

$$(\mathbf{p}-\mathbf{a})\cdot\mathbf{u}\,\mathbf{B}-(\mathbf{p}-\mathbf{a})\mathbf{u}\cdot\mathbf{B}=(\mathbf{b}-\mathbf{a})\wedge\mathbf{B}\,\mathbf{u}\,. \tag{9b}$$

But from (3), we get that

$$(\mathbf{p}-\mathbf{a})\mathbf{u}\mathbf{B}-(\mathbf{p}-\mathbf{a})\mathbf{u}\cdot\mathbf{B}=(\mathbf{p}-\mathbf{a})\mathbf{u}\wedge\mathbf{B}=(\mathbf{b}-\mathbf{a})\wedge\mathbf{B}\,\mathbf{u}\,, \tag{10}$$

where we used that

$$\mathbf{u}\mathbf{B}-\mathbf{u}\cdot\mathbf{B}=\mathbf{u}\wedge\mathbf{B}\,. \tag{11}$$

But since $\mathbf{u}\wedge\mathbf{B}$ is a pseudoscalar, we can just divide through (10) by it and then add $\mathbf{a}$ to both sides, to get

$$\mathbf{p}=\mathbf{a}+\frac{(\mathbf{b}-\mathbf{a})\wedge\mathbf{B}}{\mathbf{u}\wedge\mathbf{B}}\,\mathbf{u}\,, \tag{12}$$

and we are finished.

References

[1] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.