Math Diversion 605

P. Reany

May 23, 2025

You cannot ask us to take sides against arithmetic.
— Winston Churchill

1 The Problem

On page 93 of NFCM [1], we find problem (6.10) to prove **Ceva's Theorem**.

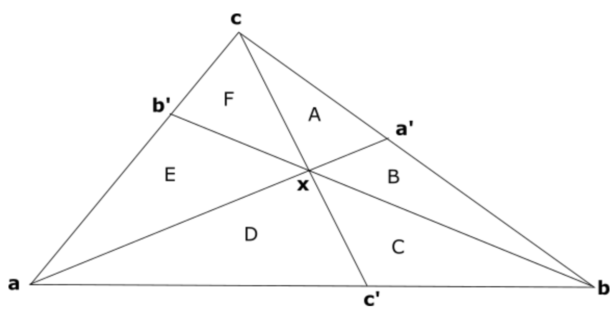

Figure 1. The triangle for Ceva's Theorem. The point $\mathbf{x}$ is the point of concurrence. The capital letters represent areas of their enclosing triangles.

Based on the above figure, show that

$$\left(\frac{\mathbf{a}-\mathbf{c}'}{\mathbf{c}'-\mathbf{b}}\right)\left(\frac{\mathbf{b}-\mathbf{a}'}{\mathbf{a}'-\mathbf{c}}\right)\left(\frac{\mathbf{c}-\mathbf{b}'}{\mathbf{b}'-\mathbf{a}}\right)=1\,. \tag{1}$$

2 Solution

We will use Eq. (6.17a) on page 85, from which we get that, relative to point $\mathbf{x}$:

$$\frac{\mathbf{a}-\mathbf{c}'}{\mathbf{c}'-\mathbf{b}}=\frac{D}{C}\,, \tag{2}$$

but relative to point $\mathbf{c}$:

$$\frac{\mathbf{a}-\mathbf{c}'}{\mathbf{c}'-\mathbf{b}}=\frac{F+E+D}{A+B+C}\,. \tag{3}$$

So, from (2) and (3), we get

$$\frac{D}{C} = \frac{F+E+D}{A+B+C}\,. \tag{4}$$

Multiplying across, we have

$$D(A+B+C) = (F+E+D)C\,. \tag{5}$$

On simplifying, we get

$$D(A+B) = (F+E)C\,. \tag{6}$$

So, now we have a new form for D/C:

$$\frac{D}{C} = \frac{F+E}{A+B}\,. \tag{7}$$

Going back to (2), we have that

$$\frac{\mathbf{a}-\mathbf{c}'}{\mathbf{c}'-\mathbf{b}} = \frac{F+E}{A+B}\,. \tag{8}$$

By analogy to the previous arguments, we have for the other two factors in (1)

$$\frac{\mathbf{b}-\mathbf{a}'}{\mathbf{a}'-\mathbf{c}} = \frac{B}{A} = \frac{C+D}{F+E}\,, \tag{9}$$

and

$$\frac{\mathbf{c}-\mathbf{b}'}{\mathbf{a}'-\mathbf{c}} = \frac{F}{E} = \frac{A+B}{D+C}\,. \tag{10}$$

Substituting into (1) gives

$$\begin{aligned}\left(\frac{\mathbf{a}-\mathbf{c}'}{\mathbf{c}'-\mathbf{b}}\right)\left(\frac{\mathbf{b}-\mathbf{a}'}{\mathbf{a}'-\mathbf{c}}\right)\left(\frac{\mathbf{c}-\mathbf{b}'}{\mathbf{b}'-\mathbf{a}}\right) &= \frac{D}{C}\,\frac{B}{A}\,\frac{F}{E} \\ &= \frac{F+E}{A+B}\,\frac{C+D}{F+E}\,\frac{A+B}{D+C} \\ &= 1\,. \end{aligned} \tag{11}$$

References

[1] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.