Math Diversion Problem 608

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May 25, 2025

> Easy to criticize, more difficult to be correct.
> — Charlie Chan

The YouTube video is found at:

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Source: The Ether of Mathematical Ideas
Title: Contrived Lambert W Function Problem #2
 Presenter: Patrick
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1 The Problem

Given the relation

$$y = x \ln x \,, \tag{1}$$

find $y' = dy/dx$.

Note: I have been patiently waiting for an opportunity to use the derivative of the Lambert W function in a real problem, but so far I haven't succeeded. So, I'm going to force it into this problem, just for the experience of using it. The experience I get here may come in handy later on.

2 The Preparation

Using the product rule on (1), we get

$$y' = \ln x + 1 \,, \tag{2}$$

so now we know what we're shooting for.

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B \,, \tag{3}$$

then

$$z = W(B) \,, \tag{4}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

Note:

$W_0(-e^{-1}) = -1$.

A lemma I'll need from the theory of the Lambert W function is the following: If

$$y \ln y = B\,, \tag{5}$$

then

$$\ln y = W(y \ln y) = W(B)\,. \tag{6}$$

The following is the 'Lambert W function base s'[1], or W_s, where s is a positive real number. Let's begin with the relation

$$xs^x = A\,, \tag{7}$$

which looks very similar to (3). Then

$$x = W_s(xs^x) \equiv \frac{W(A \ln s)}{\ln s}\,. \tag{8}$$

But when $s = e$, we have that

$$x = W_e(xe^x) = \frac{W(A \ln e)}{\ln e} = W(A)\,, \tag{9}$$

which is the usual Lambert W function. (By the way, the proof to this lemma is not hard. It begins with setting $s^x = e^y$ and proceeding from there.)

If s is an integer, I may resort to putting parentheses around it to distinguish it from the n-series, as such $W_{(s)}$.

By the way, the derivative of $W(z)$ by z is:

$$\frac{dW(z)}{dz} = \frac{W(z)}{z(1 + W(z))}\,, \tag{10}$$

where $z \neq 0$ and $z \neq -1/e$.

3 The Solution

On taking the Lambert W function across (1), we get

$$W(y) = \ln x\,. \tag{11}$$

[1]This notation I invented myself.

Next, we differentiate by x, getting:

$$\frac{dW(y)}{dx} = \frac{dW(y)}{dy} y' = \frac{1}{x} \,. \tag{12}$$

Hence

$$\begin{aligned} y' &= \frac{1/x}{\dfrac{dW(y)}{dy}} && \text{(13a)} \\ &= \frac{1}{x} \frac{y(1+W(y))}{W(y)} && \text{(13b)} \\ &= \frac{y}{x}[(W(y)^{-1}+1] && \text{(13c)} \\ &= \frac{x \ln x}{x} \left(\frac{1}{\ln x} + 1 \right) && \text{(13d)} \\ &= \ln x + 1 \,, && \text{(13e)} \end{aligned}$$

which is what we needed to show.