

Math Diversion Problem 613

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Learning is a treasure that will follow its owner everywhere.
— Chinese proverb

The YouTube video is found at:

Source: <https://www.youtube.com/watch?v=iPN91UfVuJg&list=PL81IATpFpPBgrG8fZ3tR041nNypY5xtEP>
Title: Math Subject GRE Practice Exam #1 GR1268
Presenter: LearnYouSomeMath

1 The Problem

Integrate the following integral:

$$\bar{I} = \int_{e^{-3}}^{e^{-2}} \frac{1}{x \log x} dx. \quad (1)$$

(A) 1 (B) $\frac{2}{3}$ (C) $\frac{3}{2}$ (D) $\log\left(\frac{2}{3}\right)$ (E) $\log\left(\frac{3}{2}\right)$

2 The Solution

Note: I rarely use integration by parts. Instead, I use something very similar to it, and a bit more general.

Note: I use I to represent an indefinite integral and \bar{I} to represent a definite integral, if there is one.

The goal of my replacement for integration by parts is to focus on the integrand, while allowing other terms to emerge with it. The hope is that the new terms will be easier to deal with than the original integrand. Once you see it in action, it should become obvious — hopefully.

Now, let's start off with the indefinite integral, and put in the limits of integration later:

$$I = \int \frac{1}{x \log x} dx. \quad (2)$$

$$D_x(\log(x \log x)) = \frac{D_x(x \log x)}{x \log x} \quad (3a)$$

$$= \frac{\log x + 1}{x \log x} \quad (3b)$$

$$= \frac{1}{x} + \frac{1}{x \log x} \quad (3c)$$

$$\cdot \quad (3d)$$

Well, so far, so good. Now, let's integrate!

$$\log(x \log x) = \int \frac{1}{x} dx + \int \frac{1}{x \log x} dx = \log x + I. \quad (4)$$

Next, we solve for I .

$$I = \log(x \log x) - \log x = \log \left| \frac{x \log x}{x} \right| = \log |\log x|. \quad (5)$$

Therefore,

$$\bar{I} = \log |\log x| \Big|_{e^{-3}}^{e^{-2}} = \log |-2| - \log |-3| = \log \left(\frac{2}{3} \right). \quad (6)$$