Math Diversion Problem 615

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> The dramatic differences between (Set, $\times$)
> and (Hilb, $\otimes$) explain why quantum
> mechanics seems weird.
> — John Baez
> (Category theory in action.)

The problem is found at:

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Source: https://www.youtube.com/watch?v=6L6dWdHC0RE
Title: Math subject GRE 3768 problem 59
Presenter:  LearnYouSomeMath
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1 The Problem

Let $\mathbb{Z}_{30}$ be the ring of integers modulo 30. Let U_{30} be the multiplicative group formed out of all the invertible elements of $\mathbb{Z}_{30}$, with identity element equal to unity. Let ϕ be a group homomorphism from U_{30} to itself, with $\ker\phi = \{1, 11\}$. Now, if $\phi(7) = 7$, which other element in U_{30} does ϕ map to 7?

(A) 11 (B) 13 (C) 17 (D) 19 (E) 29

2 Preparation

Of course ϕ is a map, but it's also a homomorphism from U_{30} to itself, which we'll probably have to make use of. So, what is it? Let G and H be groups, not necessarily distinct, and let ϕ be homomorphism from G to H. Then ϕ has the property that for all $x, y \in G$,

$$\phi(xy) = \phi(x)\phi(y)\,. \tag{1}$$

Also, the $\ker\phi$ is the set of elements in G that get mapped to the identity of H, which in the present case is the unity element.

3 Solution

The first order of business is to discover the elements in U_{30}:

$$U_{30} = \{1, 7, 11, 13, 17, 19, 23, 29\} . \tag{2}$$

Of course, each of these elements is relatively prime to 30, which guarantees that they'll have a multiplicative inverse in U_{30}.

Perhaps we can draw a hint from the cosets of U_{30} by $\ker \phi$: the coset that contains 7 also contains 17.

Now, since we know the image of both 7 and 11 under ϕ, why don't we use them together with the homomorphism property?

$$\phi(7 \cdot 11) = \phi(7)\phi(11) = \phi(7) = 7 . \tag{3}$$

But $7 \cdot 11 = 77 \equiv 17 \mod 30$. Therefore,

$$\phi(17) = 7 , \tag{4}$$

so we have our answer: (C).