Math Diversion 618

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May 29, 2025

Our greatest weakness lies in giving up. The most
certain way to succeed is always to try
just one more time.
—Thomas Edison

1 The Problem

On page 117 of NFCM [1], we find problem (8.2b): Show that

$$\mathbf{a} \cdot \nabla \hat{\mathbf{r}} = \frac{\hat{\mathbf{r}}\hat{\mathbf{r}} \wedge \mathbf{a}}{r}\,, \tag{1}$$

where

$$\mathbf{r} = \mathbf{x} - \mathbf{x}' \quad \text{and} \quad r = |\,\mathbf{x} - \mathbf{x}'\,|\,. \tag{2}$$

2 Lemma 1

$$\mathbf{a} \cdot \nabla\, \mathbf{x} = \mathbf{a}\,. \tag{3}$$

Proof: It's not often in geometric algebra to find a need to introduce a basis for a proof, but this time it will be useful. Let $\mathbf{x} = x_i \sigma_i$ (sum on the index i). Then $\mathbf{a} \cdot \nabla = a_j \partial_j$ (sum on the index j). So,

$$\mathbf{a} \cdot \nabla\, \mathbf{x} = a_j \partial_j x_i \sigma_i = a_j \delta_{ij} \sigma_i = a_j \sigma_j = \mathbf{a}\,. \tag{4}$$

Corollary: With $\mathbf{x}'$ being independent of $\mathbf{x}$ then

$$\mathbf{a} \cdot \nabla\, \mathbf{r} = \mathbf{a} \cdot \nabla\, (\mathbf{x} - \mathbf{x}') = \mathbf{a} \cdot \nabla\, \mathbf{x} = \mathbf{a}\,. \tag{5}$$

3 Lemma 2

$$\mathbf{a} \cdot \nabla\, r = \mathbf{a} \cdot \nabla \left|\mathbf{x} - \mathbf{x}'\right| = \frac{\mathbf{a} \cdot \mathbf{r}}{\left|\mathbf{x} - \mathbf{x}'\right|} = \frac{\mathbf{a} \cdot \mathbf{r}}{r} = \mathbf{a} \cdot \hat{\mathbf{r}}\,. \tag{6}$$

Proof left to the reader.

4 Solution

We start with

$$\begin{aligned}
\mathbf{a} \cdot \nabla \hat{\mathbf{r}} &= \mathbf{a} \cdot \nabla\, r^{-1}\mathbf{r} \\
&= (\mathbf{a} \cdot \nabla\, r^{-1})\mathbf{r} + r^{-1}\mathbf{a} \cdot \nabla\, \mathbf{r} \\
&= \frac{-\mathbf{a} \cdot \hat{\mathbf{r}}}{r^2}\mathbf{r} + r^{-1}\mathbf{a} \\
&= \frac{\mathbf{a} - \mathbf{a} \cdot \hat{\mathbf{r}}\, \hat{\mathbf{r}}}{r} \\
&= \frac{\hat{\mathbf{r}}\, \hat{\mathbf{r}} \wedge \mathbf{a}}{r}\,. \qquad (7)
\end{aligned}$$

References

[1] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.