Math Diversion Problem 619

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The ignorant always loud in argument.
— Charlie Chan

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=0J5PcbB8lNI
Title: A Nice and EZ Equation | Problem 538
 Presenter: aplusbi
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1 Problem

Given the relation

$$z(z+i) = \overline{z}\,, \tag{1}$$

find the nonzero complex values of z.

2 Solution

If we multiply (1) through by $\overline{z}$, we get

$$r^2(z+i) = \overline{z}^2\,. \tag{2}$$

Then, if we take the complex conjugate of (1), we have that

$$\overline{z}(\overline{z}-i) = z\,, \tag{3}$$

and then multiply (3) through by z, we get

$$r^2(\overline{z}-i) = z^2\,. \tag{4}$$

So, if we multiply (2) and (4) together, we get

$$r^4(z+i)(\overline{z}-i) = r^4\,, \tag{5}$$

or

$$(z+i)(\overline{z}-i) = 1\,. \tag{6}$$

Expanding this last equation, we get

$$z\overline{z} - i(z-\overline{z}) = 0\,, \tag{7}$$

which simplifies down to

$$r^2 = -2b\,. \tag{8}$$

On expanding (2) into its real and imaginary parts, we have that

$$r^2 a = a^2 - b^2\,, \tag{9a}$$

$$r^2(b+1) = -2ab\,. \tag{9b}$$

We can eliminate r^2 from these by use of (8), to get

$$-2ab = a^2 - b^2\,, \tag{10a}$$

$$b + 1 = a\,. \tag{10b}$$

On eliminating a between these, we get

$$2b^2 + 4b + 1 = 0\,, \tag{11}$$

whose solution is

$$b = -1 \pm \tfrac{1}{2}\sqrt{2}\,, \tag{12}$$

and this gives us the solution for a as

$$a = \pm \tfrac{1}{2}\sqrt{2}\,. \tag{13}$$

Therefore, the solution for z is

$$z = \pm \tfrac{1}{2}\sqrt{2} + (-1 \pm \tfrac{1}{2}\sqrt{2})i\,. \tag{14}$$

3 Conclusion

I solved this problem in the fashion that I usually do, but this problem perhaps should be solved by immediately substituting $z = a + bi$ into (1). If you follow this advice, you won't have to introduce r^2 into the equation. On the other hand, this route seems like it will introduce equations of degree higher that quadratic.